AN EXPLICIT $\overline{\partial}$-INTEGRATION FORMULA FOR WEIGHTED HOMOGENEOUS VARIETIES

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Abstract. Let $\Sigma$ be a weighted homogeneous (singular) subvariety of $C^n$. The main objective of this paper is to present an explicit integral formula for solving the $\overline{\partial}$-equation $\lambda = \overline{\partial}g$ on the regular part of $\Sigma$, where $\lambda$ is a $\overline{\partial}$-closed $(0,1)$-form with compact support. This formula will then be used to give Hölder estimates for the solution in case $\Sigma$ is homogeneous with an isolated singularity and $\lambda$ is bounded. A slight modification of the formula also gives an $L^2$-bounded solution operator in case $\Sigma$ is pure dimensional and homogeneous (but with arbitrary singular locus).

1. Introduction

As it is well known, solving the $\overline{\partial}$-equation forms a main part of complex analysis, but also has deep consequences on algebraic geometry, partial differential equations and other areas. In general, it is not easy to solve the $\overline{\partial}$-equation. The existence of solutions depends mainly on the geometry of the variety on which the equation is considered. There is a vast literature about this subject on smooth manifolds, both in books and papers [10, 11, 12], but the theory on singular varieties has been developed only recently.

Let $\Sigma$ be a singular subvariety of the space $C^n$, and $\lambda$ a bounded $\overline{\partial}$-closed differential form on the regular part of $\Sigma$. Fornæss, Gavosto and Ruppenthal have proposed a general technique for solving the $\overline{\partial}$-equation $\lambda = \overline{\partial}g$ on the regular part of $\Sigma$, which they have applied successfully to varieties of the form $\{z^m = w_1^{k_1} \cdots w_{n-1}^{k_{n-1}} \} \subset C^n$; see [9, 6] and [14]. They exploit the fact that such a variety can be considered as an $m$-sheeted analytic covering of $C^{n-1}$, project the problem by use of symmetric combinations to that complex number space, solve the $\overline{\partial}$-equation with certain weights in $C^{n-1}$, and construct the function $g$ from the pull-back of such solutions. There is a certain chance for this strategy to work in general because any locally

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irreducible complex space can be represented locally as a finitely sheeted analytic covering over a complex number space.

On the other hand, Acosta, Solís and Zeron have developed an alternative technique for solving the $\overline{\partial}$-equation (if $\lambda$ is bounded) at all kinds of isolated singularities of hypersurfaces in $\mathbb{C}^3$ (i.e. rational double points); see [1, 2] and [18]. They use the fact that all such varieties can be represented as quotient varieties in order to pull-back the problem into a complex number space, and to solve the equation also by use of symmetric combinations. This strategy has the drawback that not all varieties admit such a representation.

However, in both cases, the main strategy is to transfer the problem to some non-singular complex space, to solve the $\overline{\partial}$-equation in this well-known situation, and to carry over the solution to the singular variety. The main objective of this paper is to present and analyse an explicit formula for calculating solutions $g$ to the $\overline{\partial}$-equation $\lambda = \overline{\partial} g$ on the regular part of the original variety $\Sigma$, where $\Sigma$ is a weighted homogeneous variety and $\lambda$ is a $\overline{\partial}$-closed $(0,1)$-differential form with compact support. We analyse the weighted homogeneous varieties, for they are a main model for classifying the singular subvarieties of $\mathbb{C}^n$. A detailed analysis of the weighted homogeneous varieties is done in Chapter 2–§4 and Appendix B of [4].

**Definition 1.** Let $\beta \in \mathbb{Z}^n$ be a fixed integer vector with strictly positive entries $\beta_k \geq 1$. A polynomial $Q(z)$ holomorphic on $\mathbb{C}^n$ is said to be **weighted homogeneous** of degree $d \geq 1$ with respect to $\beta$ if the following equality holds for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$:

$$Q(s^\beta \ast z) = s^d Q(z),$$

with the action:

$$s^\beta \ast (z_1, z_2, \ldots, z_n) := (s^{\beta_1} z_1, s^{\beta_2} z_2, \ldots, s^{\beta_n} z_n).$$

An algebraic subvariety $\Sigma$ in $\mathbb{C}^n$ is said to be **weighted homogeneous** with respect to $\beta$ if $\Sigma$ is the zero locus of a finite number of weighted homogeneous polynomials $Q_k(z)$ of (maybe different) degrees $d_k \geq 1$, but all of them with respect to the same fixed vector $\beta$.

Let $\Sigma \subset \mathbb{C}^n$ be any subvariety. We use the following notation along this paper. The regular part $\Sigma^* = \Sigma_{reg}$ is the complex manifold consisting of the regular points of $\Sigma$, and it is always endowed with the induced metric so that $\Sigma^*$ is a Hermitian submanifold in $\mathbb{C}^n$ with corresponding volume element $dV_\Sigma$ and induced norm $\| \cdot \|_\Sigma$ on the Grassmannian $\Lambda T^* \Sigma^*$. Thus, any Borel-measurable $(0,1)$-form $\lambda$ on $\Sigma^*$ admits a representation $\lambda = \sum_k f_k d\overline{z}_k$, where the coefficients $f_k$ are Borel-measurable functions on $\Sigma^*$ which satisfy the inequality $|f_k(w)| \leq |\lambda(w)|_\Sigma$ for all points $w \in \Sigma^*$ and indexes $1 \leq k \leq n$. Notice that such a representation is by no means unique. We refer to Lemma 2.2.1 in [14] for a more detailed treatment of that point. We also introduce the $L^2$-norm of a measurable $(p, q)$-form $\mathcal{N}$ on an open set $U \subset \Sigma^*$ via the formula:

$$\| \mathcal{N} \|_{L^2_{p,q}(U)} := \left( \int_U |\mathcal{N}|_\Sigma^2 dV_\Sigma \right)^{1/2}.$$
We can now present the main result of this paper. We assume that the \( \overline{\partial} \)-differentials are calculated in the sense of distributions, for we work with Borel-measurable functions.

**Theorem 2 (Main).** Let \( \Sigma \) be a weighted homogeneous subvariety of \( \mathbb{C}^n \) with respect to a given vector \( \beta \in \mathbb{Z}^n \), where \( n \geq 2 \) and all entries \( \beta_k \geq 1 \). Consider a \((0,1)\)-form \( \lambda \) given by \( \sum f_k dw_k \), where the coefficients \( f_k \) are all Borel-measurable functions in \( \Sigma \), and \( z_1, ..., z_n \) are the Cartesian coordinates of \( \mathbb{C}^n \). Let \( \rho \in \mathbb{C} \) be fixed. The following function is well defined for almost all \( z \in \Sigma \) if the form \( \lambda \) is essentially bounded and has compact support in \( \Sigma \):

\[
(3) \quad g_\rho(z) := \sum_{k=1}^n \beta_k \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta \ast z) \frac{(w^\beta_k z_k) dw \wedge d\overline{w}}{|w|^2 (w - \rho)}.
\]

If \( \Sigma^* \) is the regular part of \( \Sigma \) and \( \lambda \) is \( \overline{\partial} \)-closed on \( \Sigma^* \setminus \{0\} \), then the function \( g_0 \) is holomorphic on \( \Sigma^* \setminus \{0\} \), and the function \( g_1 \) is a solution of the \( \overline{\partial} \)-equation \( \lambda = \overline{\partial}g_1 \) on \( \Sigma^* \setminus \{0\} \).

Note that the origin of \( \mathbb{C}^n \) is in general a singular point of \( \Sigma \) according to Definition 1 so that \( \Sigma^* \setminus \{0\} \) coincides with \( \Sigma^* \). We shall prove Theorem 2 in Section 2 of this paper. The main point of the proof is to show that \( \lambda = \overline{\partial}g_1 \) if \( \lambda \) is \( \overline{\partial} \)-closed. That is a local statement. So, we will cover \( \Sigma \) by charts which we call generalized cones. When we blow up these cones to complex manifolds, we realize that the integral formula (3) looks essentially like the inhomogeneous Cauchy-Pompeiu integral formula in one complex variable (see (15)), and we can deduce the statement by use of classical results.

The functions \( g_\rho \) defined in (3) have many interesting properties. For example, it follows easily that:

\[
(4) \quad g_1(z) - g_0(z) = \sum_{k=1}^n \beta_k \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta \ast z) \frac{(w^\beta_k z_k) dw \wedge d\overline{w}}{|w|^2 (w - 1)}.
\]

The differential \( \overline{\partial}(g_1 - g_0) = \lambda \) on \( \Sigma^* \setminus \{0\} \) if \( \lambda \) is \( \overline{\partial} \)-closed; \( [g_1 - g_0](0) = 0 \).

The change of variables \( w = su \) in (3) yields some useful identities: for every point \( z \in \Sigma \) and number \( s \neq 0 \) in \( \mathbb{C} \),

\[
(5) \quad g_s(z) = g_1(s^\beta \ast z) \quad \text{and} \quad g_0(z) = g_0(s^\beta \ast z).
\]

On the other hand, recalling some main principles of the proof, we also deduce anisotropic Hölder estimates for the \( \overline{\partial} \)-equation in the case where \( \Sigma \) is a homogeneous variety with only one isolated singularity at the origin. The main point is to use the regularity properties of the Cauchy-Pompeiu formula. We obviously need to specify the metric on \( \Sigma \): Given a pair of points \( z \) and \( w \) in \( \Sigma \), we define \( \text{dist}_{\Sigma}(z,w) \) to be the infimum of the length of piecewise smooth curves connecting \( z \) and \( w \) inside \( \Sigma \). It is clear that such curves exist in this situation, and that the length of each curve can be measured in the regular part \( \Sigma^* \) or the ambient space \( \mathbb{C}^n \), but both measures coincide, for \( \Sigma^* \) carries the induced norm. The main result of the section 3 is the following estimate:
Theorem 3 (Hölder). In the situation of Theorem 2, suppose that $\Sigma$ is homogeneous (a cone) and has got only one singularity at the origin of $\mathbb{C}^n$, so that each entry $\beta_k = 1$ in Definition 1. Moreover, assume that the support of the form $\lambda$ is contained in a ball $B_R$ of radius $R > 0$ and centered at the origin. Then, for each parameter $0 < \theta < 1$, there exists a constant $C_{\Sigma}(R, \theta) > 0$ which does not depend on $\lambda$ such that the following inequality holds for the function $g_1$ given in (3) and almost all points $z$ and $w$ in the intersection $B_R \cap \Sigma$,

$$|g_1(z) - g_1(w)| \leq C_{\Sigma}(R, \theta) \cdot \text{dist}_{\Sigma}(z, w)^\theta \cdot \|\lambda\|_{\infty}. \quad (6)$$

The notation $\|\lambda\|_{\infty}$ stands for the essential supremum of $|\lambda(\cdot)|_{\Sigma}$ on $\Sigma$; recall that $\lambda$ is bounded and has compact support. We should mention that Theorem 3 is a significant improvement of the known results about Hölder regularity. Consider for example $\{z^2 = w_1w_2\} \subset \mathbb{C}^3$. For this variety, Acosta, Fornæss, Gavosto, Solis and Zeron were only able to prove the statement of Theorem 3 for each $\theta < 1/2$ (see [1, 2, 6]); Ruppenthal obtained it in [14] also for $\theta = 1/2$ which is both far from the result in the present paper. Theorem 3 is proved in Section 3 where we will consider the difference $g = g_1 - g_0$ (see (4)) instead of $g_1$. We can make this reduction, because the fact that $\Sigma \setminus \{0\}$ is a homogeneous complex manifold yields that the holomorphic function $g_0$ in (3) is in fact constant: Equation (5) implies that $g_0(z)$ is equal to $g_0(sz)$ for every $s \neq 0$ in $\mathbb{C}$ (recall that each entry $\beta_k = 1$), so that $g_0$ is constant on all the complex lines $\mathbb{C}^* \setminus \{0\}$ passing through the origin. It follows that $g_0$ is constant on $\Sigma$, because it is holomorphic and thus constant on the compact projective manifold $\tilde{\Sigma}$ associated to $\Sigma$ in $\mathbb{CP}^{n-1}$.

Finally, similar techniques and a slight modification of equation (3) can also be used to produce a $\overline{\partial}$-solution operator with $L^2$-estimates on homogeneous subvarieties with arbitrary singular locus:

Theorem 4 ($L^2$-Estimates). Let $\Sigma$ be a pure $d$-dimensional homogeneous subvariety of $\mathbb{C}^n$, where $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Consider a $(0,1)$-form $\lambda$ given by $\sum_k f_k \overline{z_k}$, where the coefficients $f_k$ are all square integrable functions on $\Sigma$, and $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. The following function is well defined for almost all $z \in \Sigma$ whenever the form $\lambda$ has compact support on $\Sigma$:

$$g(z) := \sum_{k=1}^n \frac{1}{2\pi i} \int_{w \in \mathbb{C}} f_k(wz) \frac{w^{d-1} \overline{z_k} dw \wedge d\overline{w}}{w - 1}. \quad (7)$$

If $\Sigma^*$ is the regular part of $\Sigma$ and $\lambda$ is $\overline{\partial}$-closed on $\Sigma^* \setminus \{0\}$, then the function $g$ is a solution of the $\overline{\partial}$-equation $\lambda = \overline{\partial} g$ on $\Sigma^* \setminus \{0\}$. Finally, assuming that the support of $\lambda$ is contained in an open ball $B_R$ of radius $R > 0$ and centered at the origin, there exists a constant $C_{\Sigma}(R, 2) > 0$ which does not depend on $\lambda$ and such that:

$$\|g\|_{L^2(\Sigma \cap B_R)} \leq C_{\Sigma}(R, 2) \cdot \|\lambda\|_{L^2_{\overline{\partial}}(\Sigma)}. \quad (8)$$
We prove this theorem in Section 4. The proof is based on an analysis of the behavior of norms under blowing up the origin and the $L^2$-regularity of the Cauchy-Pompeiu formula. We should mention that - to our knowledge - an $L^2$-solution operator for forms with compact support was known before only for isolated singularities (see [8], Proposition 3.1).

The obstructions to solving the $\partial$-equation with $L^2$-estimates on subvarieties of $\mathbb{C}^n$ are not completely understood in general. An $L^2$-solution operator (for forms with non-compact support) is only known in the case where $\Sigma$ is a complete intersection\(^1\) of pure dimension $\geq 3$ with isolated singularities only. This operator was built by Fornæss, Øvrelid and Vassiliadou in [8] via an extension theorem for $\partial$-cohomology groups originally presented by Scheja [17]. Usually, the $L^2$-results come with some obstructions to the solvability of the $\partial$-equation. Different situations have been analysed in the works of Diederich, Fornæss, Øvrelid, Ruppenthal and Vassiliadou: it is shown that the $\partial$-equation is solvable with $L^2$-estimates for forms lying in a closed subspace of finite codimension of the vector space of all the $\partial$-closed $L^2$-forms if the variety has isolated singularities only [3, 5, 8, 13, 15]. Besides, in the paper [7], the $\partial$-equation is solved locally with some weighted $L^2$-estimates for forms which vanish to a sufficiently high order on the (arbitrary) singular set of the given varieties.

There is a second line of research about the $\partial$-operator on complex projective varieties. Though that area has clearly a lot in common with the topic of $\partial$ on analytic subvarieties of $\mathbb{C}^n$, it is a somewhat different theory because of the strong global tools (like Serre duality) which cannot be used in the (local) situation of Stein spaces (due to the lack of compactness).

Since the estimates in Theorem 3 and 4 are given only for homogeneous varieties, we finally propose in Section 5 of this paper a useful technique for generalizing the estimates in Theorem 3 and 4, so as to consider weighted homogeneous subvarieties instead of homogeneous ones. We do not elaborate that in detail because it is more or less straight forward and it is not clear if the results would be optimal in that case.

2. Proof of Main Theorem

Let $\{Q_k\}$ be the set of polynomials on $\mathbb{C}^n$ which defines the algebraic variety $\Sigma$ as its zero locus. The definition of weighted homogeneous varieties implies that the polynomials $Q_k(z)$ are all weighted homogeneous with respect to the same fixed vector $\beta$. Equation (1) automatically yields that every point $s^\beta \ast z$ lies in $\Sigma$ for all $s \in \mathbb{C}$ and $z \in \Sigma$, and so each coefficient $f_k(\cdot)$ in equations (3) and (4) is well evaluated in $\Sigma$. Moreover, fixing any point $z \in \Sigma$, the given hypotheses imply that the following Borel-measurable functions are all bounded and have compact support in $\mathbb{C}$,

$$w \mapsto f_k(w^\beta \ast z).$$

\(^1\)More precisely: a Cohen-Macaulay space.
Hence, the function \( g_\rho(z) \) in (3) is well defined for almost all \( z \in \Sigma \). We shall prove that \( g_1(z) \) is a solution of the equation \( \overline{\partial}g_1 = \lambda \), if the \((0,1)\)-form \( \lambda \) is \( \overline{\partial} \)-closed. We may suppose, without loss of generality and because of the given hypotheses, that the regular part of \( \Sigma \) does not contain the origin. Let \( \xi \neq 0 \) be any fixed point in the regular part of \( \Sigma \). We may suppose by simplicity that the first entry \( \xi_1 \neq 0 \), and so we define the following mapping \( \eta : \mathbb{C}^n \rightarrow \mathbb{C}^n \) and variety \( Y \),

\[
\eta(y) := \left( y_1/\xi_1 \right)^{\beta} \cdot \left( \xi_1, y_2, y_3, \ldots, y_n \right), \quad \text{for} \quad y \in \mathbb{C}^n,
\]

\[
Y := \{ \tilde{y} \in \mathbb{C}^{n-1} : Q_k(\xi_1, \tilde{y}) = 0, \forall k \}.
\]

The action \( s^\beta \cdot z \) was given in (2). We have that \( \eta(\xi) = \xi \), and that the following identities hold for all \( s \in \mathbb{C} \) and \( \tilde{y} \in \mathbb{C}^{n-1} \) (recall equation (1) and the fact that \( \Sigma \) is the zero locus of the polynomials \( \{ Q_k \} \)):

\[
Q_k(\eta(s, \tilde{y})) = (s/\xi_1)^{d_k} Q_k(\xi_1, \tilde{y}), \quad \text{and so}
\]

\[
\eta(C^* \times Y) = \{ z \in \Sigma : z_1 \neq 0 \}.
\]

The symbol \( C^* \) stands for \( \mathbb{C} \setminus \{0\} \). The mapping \( \eta \) is locally a biholomorphism whenever the first entry \( y_1 \neq 0 \). Whence, the point \( \xi \) lies in the regular part of the variety \( C^* \times Y \), because \( \xi = \eta(\xi) \) also lies in the regular part of \( \Sigma \) and \( \xi_1 \neq 0 \). Thus, we can find a biholomorphism

\[
\pi = (\pi_2, \ldots, \pi_n) : U \rightarrow Y \subset \mathbb{C}^{n-1}
\]

defined from an open domain \( U \) in \( \mathbb{C}^{d-1} \) onto an open set in the regular part of \( Y \), such that \( \pi(\zeta) \) is equal to \( (\xi_2, \ldots, \xi_n) \) for some \( \zeta \in U \). Consider the following holomorphic mapping defined for all points \( s \in \mathbb{C} \) and \( x \in U \),

\[
\Pi(s, x) := s^\beta \cdot (\xi_1, \pi(x)) = \eta(s\xi_1, \pi(x)) \in \Sigma.
\]

The image \( \Pi(\mathbb{C} \times U) \) will be called a \textbf{generalized cone} from now on. Notice that \( \Pi(C^* \times U) \) lies in the regular part of \( \Sigma \), for \( \pi(U) \) is contained in the regular part of \( Y \). The mapping \( \Pi(s, x) \) is locally a biholomorphism whenever \( s \neq 0 \), because \( \eta \) is also a local biholomorphism for \( y_1 \neq 0 \). Finally, the image \( \Pi(1, \zeta) \) is equal to \( \xi \). Hence, recalling the differential form \( \lambda \) and the function \( g_1 \) defined in (3), we only need to prove that the pull-back \( \Pi^*\lambda \) is equal to \( \overline{\partial}(g_1 \circ \Pi) = \overline{\partial}g_1 \) inside \( C^* \times U \) in order to conclude that the \( \overline{\partial} \)-equation \( \lambda = \overline{\partial}g_1 \) holds in a neighbourhood of \( \xi \). Consider the following identity obtained by inserting (2) and (11) into (3) (we define \( \pi_1(x) \equiv \xi_1 \)):

\[
g_1(\Pi(s, x)) = \sum_{k=1}^{n} \frac{\beta_k}{2\pi i} \int_C f_k(\Pi(ws, x)) \frac{\overline{w^{d_k}}}{{\pi_k}(x)} dw \wedge \overline{dw}. \tag{12}
\]

The given hypotheses on \( \lambda \) yields that the pull-back \( \Pi^*\lambda \) is \( \overline{\partial} \)-closed and bounded in \( C^* \times U \), and so it is also bounded and \( \overline{\partial} \)-closed in \( C \times U \); see Lemma 4.3.2 in [14] or Lemma (2.2) in [18]. We can use equations (2)
and (11) in order to calculate $\Pi^*\lambda$ when $\lambda$ is given by $\sum_k f_k d\bar{x}_k$, 

$$
\Pi^*\lambda = F_0(s, x) ds + \sum_{j \geq 1} F_j(s, x) d\bar{x}_j, \quad \text{with}
$$

(13) $F_0(s, x) = \sum_{k=1}^n f_k(\Pi(s, x)) \beta_k s^{\beta_k - 1} \pi_k(x)$, 

(14) $F_j(s, x) = \sum_{k=1}^n f_k(\Pi(s, x)) s^{\beta_k} \frac{\partial \pi_k}{\partial x_j}$.

Recall that $\pi_1(x) \equiv \xi_1$. Equation (11) and the fact that $\lambda$ has compact support on $\Sigma$ also imply that the previous function $F_0(s, x)$ has compact support on every complex line $\mathbb{C} \times \{x\}$ for all $x \in U$. Hence, the Cauchy-Pompeiu integral (applied to $\Pi^*\lambda$) solves the $\partial$-equation $\Pi^*\lambda = \partial G$ in the product $\mathbb{C} \times U$ if we define 

(15) $G(s, x) := \frac{1}{2\pi i} \int_{u \in \mathbb{C}} F_0(u, x) \frac{du \wedge d\bar{u}}{u - s}$ 

for $s \in \mathbb{C}$ and $x \in U$. Finally, equations (12) and (15) are identical when $s \neq 0$, for we only need to apply the change of variables $u = ws$. Thus, the differential $\partial \Pi^*g_1$ (resp. $\partial g_1$) is equal to the form $\Pi^*\lambda$ (resp. $\lambda$) in the product $\mathbb{C}^* \times U$ (resp. an open neighbourhood of $\xi$); and so, the $\partial$-equation $\lambda = \partial g_1$ holds in the regular part of $\Sigma$, because $\xi \neq 0$ was chosen in an arbitrary way in the regular part of $\Sigma$.

We conclude this section by showing that the function $g_0$ in (3) is holomorphic on $\Sigma^* \setminus \{0\}$. The previous condition is equivalent to prove that $\Pi^*g_0$ is holomorphic on $\mathbb{C}^* \times U$. We easily have that $\Pi^*g_0$ is constant with respect to the first entry $s \in \mathbb{C}^*$, because:

$$
g_0(\Pi(s, x)) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{\mathbb{C}} f_k(\Pi(ws, x)) \frac{(ws)^{\beta_k} \pi_k(x) dw \wedge d\bar{w}}{|w|^2} = \frac{1}{2\pi i} \int_{\mathbb{C}} F_0(u, x) \frac{du \wedge d\bar{u}}{u}.
$$

We have made the change of variables $u = ws$ and used equation (13). Besides, we may also calculate the derivatives with respect to $\bar{x}_j$, using the fact that $\partial \lambda = 0$ and equation (14),

$$
\frac{\partial g_0(\Pi(s, x))}{\partial \bar{x}_j} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial F_0(u, x)}{\partial \bar{x}_j} \cdot \frac{du \wedge d\bar{u}}{u} = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial F_j(u, x)}{d\bar{u}} \cdot \frac{du \wedge d\bar{u}}{u} = F_j(0, x) = 0.
$$

Previous derivations are all calculated in the sense of distributions. Nevertheless, the fact that they all vanish is sufficient to assure that $\Pi^*g_0$ is holomorphic with respect to $x$; see [10, p. 12]. Hence, the function $g_0$ is
holomorphic on a neighborhood of the arbitrary point $\xi \neq 0$ in $\Sigma^*$, because $\Pi^* g_0$ is also constant with respect to the first entry $s$.

3. HÖLDER ESTIMATES

In this section, we will prove anisotropic Hölder estimates on the subvariety $\Sigma \subset \mathbb{C}^n$ in the particular case when $\Sigma$ is homogeneous (a cone) and has got only one isolated singularity at the origin (Lemma 5). These estimates lead easily to optimal Hölder estimates on such varieties (Theorem 3). We will show later (in Section 5) how we can use previous results in order to deduce Hölder estimates on weighted homogeneous varieties with an isolated singularity, as well.

The given hypotheses imply that $\Sigma \setminus \{0\}$ is a homogeneous complex manifold in $\mathbb{C}^n$. Consider the holomorphic function $g_0$ defined on $\Sigma \setminus \{0\}$ by (3) with $\rho = 0$. Equation (5) yields that $g_0(z)$ is equal to $g_0(sz)$ for every $s \neq 0$ in $\mathbb{C}$ because all entries $\beta_k = 1$, so that $g_0$ is constant on all the complex lines $\mathbb{C}^*$ of $\Sigma \setminus \{0\}$ passing through the origin. Hence, the function $g_0$ is constant on $\Sigma$, for it is well defined, holomorphic and constant on the compact projective manifold $\tilde{\Sigma}$ associated to $\Sigma$ in $\mathbb{CP}^{n-1}$. Previous facts imply that we just need to show the Hölder estimate (6) for the function defined in (4):

$$g(z) := g_1(z) - g_0 = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{t \in \mathbb{C}} f_k(tz) \frac{\bar{z}_k dt \wedge dt}{t(t-1)}.$$  

We shall see later than it is easier to work with the function $g(z)$ above instead of $g_1(z)$. In particular, note that $g(0) = 0$.

We divide the proof of Theorem 3 into four parts. First, we will reduce the problem to suitable cones which cover $\Sigma$. Then, we give Hölder estimates for two points in the same line through the origin, followed by estimates for two points which lie in the same slice of such a suitable cone. Finally, we will combine both kinds of estimates to anisotropic Hölder estimates (Lemma 5) and deduce the statement of Theorem 3.

3.1. Reduction to Suitable Cones. Consider the compact link $K$ obtained by intersecting $\Sigma$ with the unit sphere $bB$ of radius $\sqrt{n}$ and center at the origin in $\mathbb{C}^n$. Note that every point $\xi \in K$ has got at least one coordinate with absolute value $|\xi_k| \geq 1$. We will follow the proof of Theorem 2.

Thus, given any point $\xi \in K$, we construct a generalized cone which contains it. For example, if the first entry $|\xi_1| \geq 1$, we build the subvariety $Y_\xi$ as in (9). Then, we consider a biholomorphism $\pi_\xi$ defined from an open set $U_\xi \subset \mathbb{C}^m$ into a neighborhood of $(\xi_2, ..., \xi_n)$ in $Y_\xi$, and the mapping $\Pi_\xi$ defined as in (11) from $\mathbb{C} \times U_\xi$ into $\Sigma$. We also restrict the domain of $\Pi_\xi$ to a smaller set $\mathbb{C} \times U''_{\xi}$, where: $U''_{\xi} \subset U'_{\xi} \subset U_{\xi}$, the open set $U''_{\xi}$ is smoothly bounded, and $\pi_\xi(U''_{\xi})$ is an open neighborhood of $(\xi_2, ..., \xi_n)$ in $Y_\xi$.

We assume furthermore that both $U'_{\xi}$ and $U''_{\xi}$ are simply connected. The
generalized cone \( \Pi_\xi(\mathbb{C} \times U_\xi''') \) obviously contains \( \xi \), as we wanted. We proceed in a similar way for any other entry \(|\xi_k| \geq 1\).

Now, since the link \( K \) is compact, we may choose finitely many (let us say \( N \)) points \( \xi^1, ..., \xi^N \) in \( K \) such that \( K \) itself is covered by their associated generalized cones \( C_j := \Pi_\xi(\mathbb{C} \times U_\xi'''') \). We assert that the analytic set \( \Sigma \) is covered by the cones \( C_j \). Let \( z \) be any point in \( \Sigma \setminus \{0\} \). It is easy to deduce the existence of \( s \in \mathbb{C}^* \) such that \( s^\beta \cdot z \) lies in \( K \); and so there exists an index \( 1 \leq j \leq N \) such that \( s^\beta \cdot z \) also lies in \( C_j \). We may suppose that the first entry \(|\xi_1^j| \geq 1 \), and that \( \Pi_\xi \) is given as in (11). Hence, there is a pair \((t, x)\) in the Cartesian product \( \mathbb{C}^* \times U_\xi'''' \) with

\[
\begin{align*}
\text{and so} \quad \Pi_\xi(t, x) &= t^\beta \cdot (\xi_1^j, \pi_\xi(x)) = \Pi_\xi(t/s, x).
\end{align*}
\]

The previous identity shows that the whole analytic set \( \Sigma \) is covered by the \( N \) generalized cones \( C_1, ..., C_N \). On the other hand, in order to prove the Hölder continuity of (6), we take a fixed parameter \( 0 < \theta < 1 \) and a pair of points \( z \) and \( w \) in the intersection of \( \Sigma \) with the open ball \( B_R \) of radius \( R > 0 \) and center at the origin in \( \mathbb{C}^n \). We want to show that there is a constant \( C_\Sigma(\theta) > 0 \) which does not depend on \( z \) or \( w \) such that:

\[
|g(z) - g(w)| \leq C_\Sigma(\theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\lambda\|_\infty.
\]

The first step is to show that we only need to verify previous Hölder inequality if the points \( z \) and \( w \) are both contained in \( B_R \cap C_j \), where \( C_j \) is a unique generalized cone defined as in the paragraphs above. Let \( \epsilon > 0 \) be a given parameter. The definition of \( \text{dist}_\Sigma(z, w) \) implies the existence of a piecewise smooth curve \( \gamma_\epsilon : [0, 1] \rightarrow \Sigma \) joining \( z \) and \( w \), i.e. \( \gamma_\epsilon(0) = z \) and \( \gamma_\epsilon(1) = w \), such that:

\[
\text{length}(\gamma_\epsilon) = \int_0^1 \|\gamma'(t)\| \, dt \leq \text{dist}_\Sigma(z, w) + \epsilon.
\]

The image of \( \gamma_\epsilon \) is completely contained in \( B_R \cap \Sigma \) because \( \Sigma \) is homogeneous (a cone). Now then, we are done if the points \( z \) and \( w \) are both contained in the same generalized cone \( C_j \). Otherwise, we run over the curve \( \gamma_\epsilon \) from \( z \) to \( w \), and pick up a finite set \( \{z_k\} \) inside \( \gamma_\epsilon \subset B_R \) such that: the initial point \( z_0 = z \), the final point \( z_N = w \), two consecutive elements \( z_j \) and \( z_{j+1} \) lie in the same generalized cone, and three arbitrary elements of \( \{z_k\} \) cannot lie in the same generalized cone. In particular, we may also suppose, without loss of generality, that: \( z_0 = z \) is in \( C_1 \), the final point \( z_N = w \) is in \( C_N \), and any other point \( z_j \) is in the intersection \( C_j \cap C_{j+1} \) for every index \( 1 \leq j < N \). So that, two consecutive points \( z_{j-1} \) and \( z_j \) lie in the same generalized cone \( C_j \cap B_R \) for each index \( 1 \leq j \leq N \). Assume for the moment that there exist constants \( C_\Sigma^j(\theta) > 0 \) such that

\[
|g(z_{j-1}) - g(z_j)| \leq C_\Sigma^j(\theta) \cdot \text{dist}_\Sigma(z_{j-1}, z_j)^\theta \cdot \|\lambda\|_\infty.
\]
for all $1 \leq j \leq N$. Then, it follows that
\[
|g(z) - g(w)| \leq \sum_{j=1}^{N} |g(z_{j-1}) - g(z_{j})| \leq \sum_{j=1}^{N} C_{\Sigma}^j(\theta) \dist_{\Sigma}(z_{j-1}, z_{j})^{\theta} \|\lambda\|_{\infty}
\]
\[
\leq C_{\Sigma}(\theta) \cdot [\dist_{\Sigma}(z, w) + \epsilon]^{\theta} \cdot \|\lambda\|_{\infty},
\]
where we have chosen $C_{\Sigma}(\theta) = \sum_j C_{\Sigma}^j(\theta)$. Since previous inequality holds for all $\epsilon > 0$, it follows that we only need to prove that the Hölder estimate (17) holds under the assumption that $z$ and $w$ are both contained in the intersection of a unique generalized cone $C_j$ with the open ball $B_R$ of radius $R > 0$ and center at the origin in $\mathbb{C}^n$. Moreover, we can suppose, without loss of generality, that $C_j$ is indeed the generalized cone given in (11).

We will now determine what the assumptions on $\lambda$ imply for $\Pi^* \lambda$. Recall the given hypotheses: The subvariety $\Sigma$ is homogeneous and has got only one isolated singularity at the origin of $\mathbb{C}^n$, so that each entry $\beta_k = 1$ in Definition 1. We fix a point $\xi$ in the link $K \subset \Sigma$, and assume that its first entry $|\xi_1| \geq 1$. The subvariety $Y$ is then given in (9), and the biholomorphism $\pi$ is defined from an open set $U \subset \mathbb{C}^m$ into a neighborhood of $(\xi_2, \ldots, \xi_n)$ in $Y$. Let $\lambda$ be a $(0,1)$-form as in the hypotheses of Theorem 2.

We may easily calculate the pull-back $\Pi^* \lambda$, with the mapping $\Pi$ given in (11) for all $s \in \mathbb{C}^m$ and $x \in U$,
\[
\Pi(s, x) = (s^{(1,\ldots,1)} \ast (\xi_1, \pi(x))) = (s\xi_1, s\pi(x)) \in \Sigma.
\]

The pull-back $\Pi^* \lambda = F_0(s, x) ds + \sum_j F_j dx_j$ satisfies:
\[
F_0(s, x) = \sum_{k=1}^{n} f_k(\Pi(s, x)) \pi_k(x), \quad \pi_1(x) \equiv \xi_1,
\]
\[
F_j(s, x) = \sum_{k=2}^{n} f_k(\Pi(s, x)) \left[ s \frac{\partial \pi_k}{\partial x_j} \right].
\]

The hypotheses of Theorem 3 yield that the support of every $f_k$ is contained in a ball of radius $R > 0$ and center at the origin. Whence, equation (18) and the fact that $|\xi_1| \geq 1$ automatically imply that each function $F_k(s, x)$ vanishes whenever $|s| > R$.

The biholomorphism $\pi$ has got a Jacobian (determinant) which is bounded from above and below (away from zero) in the compact closure $\overline{U}_\xi^\theta$. Hence, there exists a constant $D_1 > 0$, such that the following identities hold for every point $(s, x)$ in $\mathbb{C} \times \overline{U}_\xi^\theta$ and each index $1 \leq j \leq m$,
\[
|F_0(s, x)| \leq D_1 \cdot \|\lambda\|_{\infty},
\]
\[
|F_j(s, x)| \leq D_1 \cdot |s| \cdot \|\lambda\|_{\infty}.
\]

We may show that the Hölder estimate (17) holds for all points $z$ and $w$ in the intersection of the generalized cone $\Pi(\mathbb{C} \times U')$ with the ball $B_R$, and so we can conclude that the same estimate holds on $B_R \cap \Sigma$. 


3.2. Hölder Estimates for Points in the Same Line. Fix the parameter $0 < \theta < 1$. We are going to analyse two different cases. Firstly, we assume there exist a point $x \in U''$ and two complex numbers $s$ and $s'$, such that $z = \Pi(s, x)$ and $w = \Pi(s', x)$. We say, in this case, that $z$ and $w$ lie in the same complex line. Equation (18) and the fact that $|\xi_1| \geq 1$ yields that $|s|$ is bounded:

$$|s| \leq |s\xi_1| \leq \|z\| < R.$$  

Define the new function $G(s, x)$ as the pull back of (16):

$$G(s, x) := g(\Pi(s, x)) = g_1(\Pi(s, x)) - g_0.$$  

We easily have that $\partial G = \Pi^*\lambda$ on $\mathbb{C}^* \times U$, because $\partial g_1 = \lambda$ and $\Pi$ is a biholomorphism; see the conclusions of Theorem 2. We may calculate $G(s, x)$ via equations (18)–(19) and the change of variables $u = st$,

$$G(s, x) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{t \in \mathbb{C}} f_k(\Pi(st, x)) \frac{s \pi_k(x) dt \wedge d\bar{t}}{t(t-1)}$$

$$= \frac{s}{2\pi i} \int_{|u| \leq R} F_0(u, x) \frac{du \wedge d\bar{u}}{u(u-s)}.$$  

Recall that $F_0(u, x)$ vanishes whenever $|u| > R$. It is then easy to see that $G(s, x)$ is bounded in $\mathbb{C} \times U$ and $G(0, x) = 0$. Hence, Lemma 4.3.2 in [14] implies that $\partial G$ is equal to $\Pi^*\lambda$ in $\mathbb{C} \times U$. We also have that:

$$|g(z) - g(w)| = |G(s, x) - G(s', x)|$$

$$= \frac{1}{2\pi} \left| \int_{|u| \leq R} F_0(u, x) \left( \frac{1}{u-s} - \frac{1}{u-s'} \right) du \wedge d\bar{u} \right|.$$  

It is well known that there exists a constant $D_2(R, \theta) > 0$, depending only on the radius $R > 0$ and the parameter $\theta$, such that:

$$|g(z) - g(w)| \leq D_2(R, \theta) |s - s'|^\theta D_1 \|\lambda\|_\infty.$$  

Note that we have used (20), and consider chapter 6.1 of [14] for a (more general) version of the inequality above. The analysis done in the previous paragraphs shows that (17) holds in the first case. Besides, since both $g(0)$ and $G(0, x)$ vanish, we also obtain the following useful estimate:

$$|G(s, x)| = |g(z)| \leq D_2(R, \theta) D_1 |s|^{\theta} \|\lambda\|_\infty.$$  

3.3. Hölder Estimates for Points in the Same Slice. Let $z$ and $\hat{w}$ be a pair of points in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball $B_R$. Assume that there exist a complex number $s \neq 0$ and a pair of points $x$ and $x'$ in the open set $U''$ such that $z = \Pi(s, x)$ and $\hat{w} = \Pi(s, x')$. We say, in this case, that $z$ and $\hat{w}$ lie in the same slice. By a unitary change of coordinates which does not destroy the inequality (21), we may assume that the entries of $x$ and $x'$ are all equal, with the possible exception of the first one. That is, we may assume that both $x$ and $x'$ lie in the complex line $L := \mathbb{C} \times \{(x_2, ..., x_m)\}$. Recall that the differential $\partial G$ is equal to $\Pi^*\lambda$ in the open
set $\mathbb{C} \times U$, according to definition (23) and the paragraphs below it. Hence, we can evaluate $g(z)$ via the inhomogeneous Cauchy-Pompeiu formula on the line $L$:

$$g(z) = G(s, x) = \frac{1}{2\pi i} \int_{L \cap U'} F_1(s, t, x_2, \ldots, x_m) \frac{dt \wedge d\bar{t}}{t - x_1} + \frac{1}{2\pi i} \int_{L \cap bU'} G(s, t, x_2, \ldots, x_m) \frac{dt}{t - x_1},$$

because $x$ is in $L \cap U''$ and $U'' \subseteq U'$. We introduce some notation in order to simplify the analysis. The symbols $I_1(s, x)$ and $I_2(s, x)$ stand for the above integrals on the set $L \cap U'$ and the boundary $L \cap bU'$, respectively. In particular, we have that:

$$g(\hat{w}) = G(s, x') = I_1(s, x') + I_2(s, x').$$

Recall that $x$ and $x'$ are both in $L \cap U''$, and that the difference $x - x'$ is equal to the vector $(x_1 - x'_1, 0, \ldots, 0)$. Inequality (21) implies the existence of a constant $D_3(\theta) > 0$, depending only on the diameter of $U''$ and the parameter $\theta$, such that:

$$|I_1(s, x) - I_1(s, x')| \leq D_3(\theta) |x_1 - x'_1|^{\theta} D_1 |s| \|\lambda\|_{\infty}. \tag{26}$$

We can calculate similar estimates for $I_2$: Let $\delta > 0$ be the distance between the compact sets $\overline{U''}$ and $bU''$ in $\mathbb{C}^m$. We obviously have that $\delta > 0$ because $U'' \subseteq U'$. The following estimates are deduced from (25) and the mean value theorem, the maximum is calculated over all $v$ in $L \cap \overline{U''}$,

$$|I_2(s, x) - I_2(s, x')| \leq \frac{|x_1 - x'_1|}{2\pi} \max_v \left| \int_{L \cap bU''} \frac{G(s, t, x_2, \ldots) dt}{(t - v)^2} \right| \tag{27}$$

$$\leq \frac{|x_1 - x'_1|}{2\pi} \frac{\text{length}(L \cap bU'')}{\delta^2} D_2(R, \theta) D_1 |s|^{\theta} \|\lambda\|_{\infty}. \tag{28}$$

### 3.4. Hölder Estimates for any two points

The estimates (24), (26) and (27), (28) can be summarized in the following lemma. It is convenient to recall that the points $x$ and $x'$ are both contained in the bounded set $U'' \subseteq \mathbb{C}^m$. Moreover, we also have that $|s| < R$ and $|s'| < R$, because $z$, $w$ and $\hat{w}$ are all contained in the ball $B_R$; recall the proof of (22).

**Lemma 5 (Anisotropic Estimates).** In the situation of Theorem 2 and Theorem 3, consider the functions $g$ and $\Pi$ given in (16) and (18), respectively, and the bounded open set $U'' \subseteq \mathbb{C}^m$ defined above. Then, for every parameter $0 < \theta < 1$, there is a constant $D_4(R, \theta) > 0$ which does not depend on $\lambda$ such that the following statements hold for all the points $z = \Pi(s, x)$ and $w = \Pi(s', x')$ in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball $B_R$:

$$|g(z) - g(w)| \leq D_4(R, \theta) |s - s'|^{\theta} \|\lambda\|_{\infty},$$

whenever $z = x'$, i.e. $z$ and $w$ are in the same line; and:

$$|g(z) - g(w)| \leq D_4(R, \theta) |x - x'|^{\theta} |s|^{\theta} \|\lambda\|_{\infty},$$

whenever $s = s'$, i.e. $z$ and $w$ are in the same slice.
It is now easy to prove that the Hölder estimates given in (6) and (17) hold for all points \( z \) and \( w \) which fulfill the assumptions of Lemma 5, namely that lie in the intersection of the generalized cone \( \Pi(\mathbb{C} \times U''') \) with the ball \( B_R \). The definition of \( \Pi \), given in (18), allows to write down the identities:

\[
z = \Pi(s, x) = s(\xi_1, \pi(x)) \quad \text{and} \quad w = \Pi(s', x') = s'(\xi_1, \pi(x')).
\]

(29)

Fix the point \( z' := \Pi(s, x') = s(\xi_1, \pi(x')) \) which is in the same line as \( w \) and in same the slice as \( z \). We can suppose, without loss of generality, that \( z' \in B_R \) because \( z \) and \( w \) also lie in \( B_R \). Otherwise, if the norm \( \|z'\| \geq R \), we only need to use \( \Pi(s', x) \) instead. We can easily deduce the following estimate from (29) and the fact that \( |\xi_1| \geq 1 \),

\[
|s - s'| \leq |s\xi_1 - s'\xi_1| \leq \|z - w\| \leq \text{dist}_\Sigma(z, w).
\]

Recall that \( \pi \) is a biholomorphism whose Jacobian (determinant) is bounded from above and below (away from zero) on the compact set \( U''' \). Hence, recalling (29), we can deduce the existence of a constant \( D_5 > 0 \), depending only on \( \pi \) and \( U''' \), such that:

\[
\frac{|s| \cdot \|x-x'\|}{D_5} \leq |s| \cdot \|\pi(x)-\pi(x')\| \leq \|z-z'\|
\leq \|z-w\| + \|w-z'\| = \|z-w\| + |s-s'| \cdot \|\xi_1, \pi(x')\|
\leq \text{dist}_\Sigma(z, w) \cdot [2 + \|\pi(x')\|].
\]

Thus, there exists a constant \( D_6 > 0 \), depending only on \( \pi \) and \( U''' \), such that the following identities hold for all the points \( z = \Pi(s, x) \) and \( w = \Pi(s', x') \) in the intersection of \( \Pi(\mathbb{C} \times U''') \) with the ball \( B_R \):

\[
|s-s'| \leq D_6 \cdot \text{dist}_\Sigma(z, w) \quad \text{and} \quad |s| \cdot \|x-x'\| \leq D_6 \cdot \text{dist}_\Sigma(z, w).
\]

Recall that \( z' \) is in the same line as \( w \) and in same the slice as \( z \); so that Lemma 5 yields that:

\[
|g(z) - g(w)| \leq |g(z) - g(z')| + |g(z') - g(w)|
\leq D_4(R, \theta) \left[ |s|^{\theta} \cdot \|x-x'\|^{\theta} + |s-s'|^{\theta} \right] \|\lambda\|_{\infty}
\leq D_4(R, \theta) 2D_6^{\theta} \text{dist}_\Sigma(z, w)^{\theta} \|\lambda\|_{\infty}.
\]

This completes the proof that the Hölder estimates given in (6) and (17) hold for all pairs of points \( z \) and \( w \) in the intersection of the ball \( B_R \) with the generalized cone \( \Pi(\mathbb{C} \times U''') \); and so we can conclude that the same Hölder estimates hold for arbitrary points \( z \) and \( w \) in \( B_R \cap \Sigma \), we just need to recall the reduction argument of subsection 3.1.

4. \( L^2 \)-Estimates

We will prove Theorem 4 in this section. So, we begin by showing that the function \( g \) given in (7) is indeed well defined for almost all \( z \in \Sigma \). Recall
that
\[
g(z) = \sum_{k=1}^{n} \frac{H_k(z)}{2\pi i}, \quad H_k(z) := \int_{|w| \leq R} f_k(wz) \frac{w^d \overline{z_k} dw \wedge d\overline{w}}{w(w-1)},
\]
and that \( \Sigma \) is a pure \( d \)-dimensional homogeneous subvariety of \( \mathbb{C}^n \), so that \( n \geq 2 \) and each entry \( \beta_k = 1 \) in Definition 1, and that the \((0,1)\)-form \( \lambda \) is given by \( \sum f_k d\overline{z_k} \), where the coefficients \( f_k \) are all square-integrable functions in \( \Sigma \). Assume that the support of \( \lambda \) is contained in the open ball \( B_R \) of radius \( R > 0 \) and center at the origin. We only need to show that the following integrals exist:
\[
\int_{w \in \mathbb{C}} \left| \int_{z \in \Sigma \cap B_R} \frac{f_k(wz) w^d \overline{z_k}}{w(w-1)} \right| dV_z dV_w < \infty.
\]
A direct application of Fubini’s theorem will yield that the integrals in (30) are all well defined for almost all \( z \in \Sigma \cap B_R \), and so they are also well defined for almost all \( z \in \Sigma \) because the radius \( R \) can be as large as we like. The fact that \( \Sigma \) is a pure 2\( d \)-real dimensional and homogeneous (cone) subvariety implies the existence of a constant \( C_0 > 0 \) such that the following equations hold for all \( w \in \mathbb{C} \) and real \( \rho > 0 \):
\[
\int_{z \in \Sigma \cap B_\rho} \|z\|^2 = C_0 \rho^{2d+2}, \quad \int_{z \in \Sigma} |f_k(wz)|^2 = \frac{\|f_k\|^2_{L^2(\Sigma)}}{|w|^{2d}}.
\]
Recall that each \( \|f_k\|^2_{L^2(\Sigma)} < \infty \) because of the given hypotheses. Note that we need not calculate the integral (31) in the Cartesian product of \( \mathbb{C} \) times \( \Sigma \cap B_R \). We can simplify the calculations by integrating over the set \( \Xi \) defined below, because \( f_k(wz) = 0 \) whenever \( |wz| \geq R \):
\[
\Xi := \{(w, z) \in \mathbb{C} \times \Sigma : |z| < R, \ |wz| < R\}.
\]
By use of (32), it follows easily that
\[
\left\| \frac{f_k(wz) w^d}{|w^2 - w|^{2/3}} \right\|^2_{L^2(\Xi)} \leq \int_{w \in \mathbb{C}} \int_{z \in \Sigma} \left| \frac{f_k(wz) w^d}{|w^2 - w|^{4/3}} \right|^2 \leq \frac{1}{|w^2 - w|^{4/3}} < \infty,
\]
and that
\[
\left\| \frac{z_k}{|w^2 - w|^{1/3}} \right\|^2_{L^2(\Xi)} \leq \int_{w \in \mathbb{C}} \int_{z \in \Sigma \cap B_R} \frac{|z|^2}{|w^2 - w|^{2/3}} \leq \frac{C_0 R^{2d+2}}{|w^2 - w|^{2/3}} + \int_{|w| > 1} \frac{C_0 (R/|w|)^{2d+2}}{|w^2 - w|^{2/3}} < \infty.
\]
The last integral in the first line of (35) must be separated into two parts depending on the value of \( |w| \). Then, one must apply (32) with \( \rho \) equal to \( R \) or \( R/|w| \), respectively.
Now, the Cauchy-Schwartz inequality \( \|ab\|_{L^1} \leq \|a\|_{L^2} \|b\|_{L^2} \) allows to
dedupe (31) from the inequalities (34) and (35), for in (31), we only need to
integrate over the set \( \Xi \) given in (33).

The next step is to show that \( g \) in (30) satisfies the differential equation
\( \overline{\partial}g = \lambda \) on \( \Sigma^* \). We only need to follow step by the step the proof presented
in Section 2. The only difference is that we must use a weighted Cauchy-
Pompeiu integral in (14), with \( m = d-1 \) integer:

\[
G(s, x) := \frac{1}{2\pi i} \int_{u \in \mathbb{C}} \frac{F_0(u, x)}{u - s} \left[ \frac{u^m}{s^m} \right] du \wedge d\overline{u},
\]

where \( F_0(u, x) = \sum_{k=1}^{n} f_k(\Pi(u, x)) \overline{\pi_k(x)} \).

Notice that \( \Pi(u, x) = u(\xi_1, \pi(x)) \) because each entry \( \beta_k = 1 \) in (2)
and (11). We must prove that \( u^m \Pi^* \lambda \) is in \( L^2_{0,1}(\mathbb{C} \times U) \) and \( \overline{\partial} \)-closed. It is
easy to calculate the pull-back of the volume form on \( \Sigma \):

\[
\Pi^*dV_{\Sigma} = \sum_{|I|=|J|=d} \beta_{I,J}(z) dz_I \wedge d\overline{z}_J \bigg|_{z=u(\xi_1, \pi(x))} = \Theta(x)|u|^{2d-2} \left[ du \wedge d\overline{u} \right] \wedge \bigwedge_{k=1}^{d-1} \left[ dx_k \wedge d\overline{x}_k \right].
\]

Recall that \( x \) lies in \( U \subset \mathbb{C}^{d-1} \). Since \( \Sigma \) is a \( d \)-dimensional homogeneous
(cone) subvariety of \( \mathbb{C}^n \), the coefficients \( \beta_{I,J}(z) \) are all invariant under the
transformations \( z \mapsto uz \), and so \( \Theta(x) \) only depends on the vales of \( \pi(x) \) and
all its partial derivatives. The fact that \( \Pi \) is a biholomorphism from \( \mathbb{C}^* \times U \)
on to its image also implies that \( \Theta \) (which is constant in \( u \)) cannot vanish.
Hence, choosing a smaller set \( U \) if it is necessary, we can suppose that \( |\Theta| \)
is bounded from below by a constant \( M > 0 \). It is easy to see that \( u^m \Pi^* \lambda \)
is \( L^2_{0,1} \), because:

\[
M \int_{\mathbb{C} \times U} |u^m \Pi^* f_k|^2 dV_{\mathbb{C} \times U} \leq \int_{\Pi(\mathbb{C} \times U)} |f_k|^2 dV_{\Sigma} \leq \|\lambda\|^2_{L^2_{0,1}(\Sigma)} < \infty.
\]

We have used equation (37) with \( m = d-1 \). Working as in Section 2,
it follows from Lemma 4.3.2 in [14] that \( u^m \Pi^* \lambda \) is \( \overline{\partial} \)-closed, and that
the differential \( \overline{\partial} G \) is equal to \( [s^m \Pi^* \lambda]/s^m = \Pi^* \lambda \). Hence, the function \( g \) given
in (30) is a solution to \( \overline{\partial} g = \lambda \) on \( \Sigma^* \), because \( g(\Pi(s, x)) \) is identically equal
to (36) after setting \( u = sw \) and \( \pi_1(x) \equiv \xi_1 \).

Finally, we must calculate the \( L^2 \)-norm of \( g \) in order to prove the \( L^2 \)-estimates (8). It is well-known that the Cauchy-Pompeiu formula is an
\( L^2 \)-bounded operator:

\[
\int_{|t|<R} \frac{1}{2\pi i} \int_{|u|<R} h(u) \frac{du \wedge d\overline{u}}{u - t} \bigg| dV_{\mathbb{R}}(t) \lesssim \int_{|t|<R} |h(t)|^2 dV_{\mathbb{C}}.
\]
The reader may find a complete proof in [12] or [14], for example. Let \( \tilde{\Sigma} \) be the projective variety associated to \( \Sigma \) in the space \( \mathbb{C}P^{n-1} \). We will use the fact that

\[
\int_{z \in \Sigma} \Phi(z) \, dV_{\Sigma}(z) = \int_{[z] \in \tilde{\Sigma}} \int_{t \in \mathbb{C}} \Phi(zt)|t|^{2d-1} \, dV_{\mathbb{C}}(t) \, dV_{\Sigma}([z])
\]

where on the right hand side, \( z \) is any representative of \( [z] \) with \( \|z\| = 1 \). It is easy to calculate each norm \( \|H_k\|_{L^2(\Sigma)} \) in (30), with \( m = d-1 \) and \( u = wt \),

\[
\begin{align*}
&\int_{z \in \Sigma \cap B_R} \int_{|w| < R} \left| \frac{f_k(wz)w^d}{w(w-1)} \right|^2 dV_{\Sigma} \\
&\leq \int_{z \in \Sigma} \int_{|t| < R} |t|^2 \left| \frac{f_k(wzt)w^m}{w-1} \right|^2 |t|^{2m} dV_{\mathbb{C}} \, dV_{\Sigma} \\
&= \int_{z \in \Sigma} \int_{|u| < R} \left( \frac{f_k(uz)u^m}{u-t} \right)^2 \left\| \frac{1}{|t|^2} \right\| |t|^{2m} dV_{\mathbb{C}} \, dV_{\Sigma} \\
&\leq \int_{z \in \Sigma} \int_{|t| < R} \left| \frac{f_k(tz)t^m}{u-t} \right|^2 dV_{\mathbb{C}} \, dV_{\Sigma} \\
&= \int_{z \in \Sigma \cap B_R} |f_k(z)|^2 dV_{\Sigma} = \|f_k\|^2_{L^2(\Sigma)} \leq \|\lambda\|^2_{L^2_{0,1}(\Sigma)}.
\end{align*}
\]

We have used equation (38) with \( h(u) = f_k(uz)u^m \). That completes the proof of Theorem 4, because:

\[
\|g\|_{L^2(\Sigma)} \leq \sum_{k=1}^n \frac{\|H_k\|_{L^2(\Sigma)}}{2\pi i} \lesssim \|\lambda\|_{L^2_{0,1}(\Sigma)}.
\]

**5. Weighted Homogeneous Estimates**

We want to close this paper presenting a useful technique for generalizing the estimates given in Theorems 3 and 4, so as to consider weighted homogeneous subvarieties instead of cones. Let \( X \subset \mathbb{C}^n \) be a weighted homogeneous subvariety with only one singularity at the origin and defined as the zero locus of a finite set of polynomials \( \{Q_k\} \). Thus, the polynomials \( Q_k(x) \) are all weighted homogeneous with respect to the same vector \( \beta \in \mathbb{Z}^n \), and each entry \( \beta_k \geq 1 \). Define the following holomorphic mapping:

\[
(39) \quad \Theta : \mathbb{C}^n \to \mathbb{C}^n, \quad \text{with} \quad \Theta(z) = (z_1^{\beta_1}, z_2^{\beta_2}, ..., z_n^{\beta_n}).
\]

It is easy to see that each polynomial \( Q_k(\Theta) \) is homogeneous, and so the subvariety \( \Sigma \subset \mathbb{C}^n \) defined as the zero locus of \( \{Q_k(\Theta)\} \) is a cone. Moreover, since \( \Theta \) is locally a biholomorphism in \( \mathbb{C}^n \setminus \{0\} \), we have that \( \Sigma \) has got only one singularity at the origin as well. Consider a \( (0,1) \)-form \( \eta \) given by the sum \( \sum_k f_k dw_k \), where the coefficients \( f_k \) are all Borel-measurable functions...
in $X$, and $x_1, \ldots, x_n$ are the Cartesian coordinates of $\mathbb{C}^n$. We may follow two different paths in order to solve the equation $\overline{\partial} h = \aleph$. We may apply the main Theorem 2, whenever $\aleph$ is bounded and has compact support on $X$, so as to get the solution:

$$h(x) = \sum_{k=1}^{n} \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w)^x_x x_k \frac{d w \wedge d\overline{w}}{w(w-1)}.$$

Otherwise, we may consider the pull-back $\Theta^* \aleph$, and apply Theorem 3, in order to solve the equation $\overline{\partial} g = \Theta^* \aleph$ on $\Sigma$. We easily have that:

$$\Theta^* \aleph = \sum_{k=1}^{n} f_k(\Theta(z)) \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(\Theta(w z)) \frac{(w z_k)^{\beta_k} d w \wedge d\overline{w}}{w(w-1)}.$$

Both paths yield exactly the same solution because $g(z)$ is identically equal to $h(\Theta(z))$. Recall that $w^\beta \Theta(z)$ is equal to $\Theta(wz)$ for all $w \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Hence, we may calculate the solution $g$ above, and use the Hölder estimates given in equation (6),

$$|g(z) - g(w)| \leq C_{\Sigma}(R, \theta) \cdot \text{dist}_{\Sigma}(z, w)^{\theta} \cdot ||\Theta^* \aleph||_{\infty}.$$

A final step is to push forward these estimates, in order to deduce similar Hölder estimates for the solution $h$ on $X$. A detailed analysis on the procedure for pushing forward the Hölder estimates can be found in [18]. On the other hand, we may use a similar procedure for $L^2$-estimates. In that case the subvariety $X$ could have arbitrary singularities.

**References**


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