FRIEDRICHS’ EXTENSION LEMMA WITH BOUNDARY VALUES AND APPLICATIONS IN COMPLEX ANALYSIS

J. RUPPENTHAL

Abstract. Let $Q$ be a first-order differential operator on a compact, smooth oriented Riemannian manifold with smooth boundary. Then, Friedrichs’ Extension Lemma states that the minimal closed extension $Q_{\text{min}}$ (the closure of the graph) and the maximal closed extension $Q_{\text{max}}$ (in the sense of distributions) of $Q$ in $L^p$-spaces ($1 \leq p < \infty$) coincide. In the present paper, we show that the same is true for boundary values with respect to $Q_{\text{min}}$ and $Q_{\text{max}}$. This gives a useful characterization of weak boundary values. As an application, we derive the Bochner-Martinelli-Koppelman formula for $L^p$-forms with weak $\partial$-boundary values.

1. Introduction

Let $M$ be a smooth, compact Riemannian manifold with smooth boundary, $E$ and $F$ Hermitian vector bundles over $M$, and $Q : C^\infty(M, E) \to C^\infty(M, F)$ a differential operator of first order. Let $1 \leq p < \infty$ and $f \in L^p(M, E)$. We say that $f \in \text{dom}(Q_{\text{min}}^p)$ if there exists a sequence $\{f_j\} \subset C^\infty(M, E)$ and a section $g \in L^p(M, F)$ such that $f_j \to f$ in $L^p(M, E)$, $Qf_j \to g$ in $L^p(M, F)$, and define $Q_{\text{min}}^p f := g$. The well-defined operator $Q_{\text{min}}^p$ is called the minimal extension of $Q$ because it is the closed extension of $Q$ to an operator $L^p(M, E) \to L^p(M, F)$ with minimal domain of definition. Its graph is simply the closure of the graph of $Q : C^\infty(M, E) \to C^\infty(M, F)$ in $L^p(M, E) \times L^p(M, F)$. Let $\sigma_Q$ be the principal symbol of $Q$, $\nu$ the outward pointing unit normal to $bM$, and $\nu^\flat$ the dual cotangent vector. Then, we say that $f$ has boundary values with respect to $Q_{\text{min}}^p$ if there exists a sequence $\{f_j\} \subset C^\infty(M, E)$ such that $\lim_{j \to \infty} f_j = f$ in $L^p(M, E)$, $\lim_{j \to \infty} Qf_j = Q_{\text{min}}^p f$ in $L^p(M, F)$, and a section $f_b \in L^p(bM, E|_{bM})$ such that

$$\lim_{j \to \infty} \sigma_Q(\cdot, \nu^\flat(\cdot))f_j|_{bM} = \sigma_Q(\cdot, \nu^\flat(\cdot))f_b \quad \text{in} \quad L^p(bM, F|_{bM}).$$

In this case, we call $f_b$ weak $Q$-boundary values of $f$ with respect to $Q_{\text{min}}^p$ (i.e. in the sense of approximation).

Date: March 3, 2008.

1991 Mathematics Subject Classification. 47F05, 32A25, 32A70.

Key words and phrases. Weak boundary values, generalized Stokes’ formulas, Cauchy-Riemann equations.
Now, we draw our extension to the maximal closed extension of $Q$, that is the extension of $Q$ in the sense of distributions. We say that $f \in \text{dom}(Q_{\text{max}}^p)$, if $Qf = u \in L^p(M, F)$ in the sense of distributions, and set $Q_{\text{max}}^p f := u$ in that case. Here again, we can define weak $Q$-boundary values with respect to $Q_{\text{max}}^p$.

We say that $f$ has weak $Q$-boundary values $f_b \in L^p(bM, E|_{bM})$ with respect to $Q_{\text{max}}^p$ (in the sense of distributions), if $f_b$ satisfies the generalized Green-Stokes formula (Theorem 2.1)

$$(Qu, \phi)_M - (u, Q^*\phi)_M = \frac{1}{i} \int_{bM} \langle \sigma_Q(x, \nu^\flat)u_b, \phi \rangle_{F_x} dS(x)$$

for all $\phi \in C^\infty(M, F)$. So, this is really a definition in the sense of distributions.

The main objective of the present paper is to compare both notions of $Q$-boundary values. It is easy to see that $\text{dom}(Q_{\text{min}}^p) \subset \text{dom}(Q_{\text{max}}^p) \subset L^p(M, E)$, and that $Q_{\text{min}}^p$ is the restriction of $Q_{\text{max}}^p$ to $\text{dom}(Q_{\text{min}}^p)$. Moreover, it is also clear that weak $Q$-boundary values in the sense of approximation are weak $Q$-boundary values in the sense of distributions, as well. It is well-known that in fact $Q_{\text{min}}^p = Q_{\text{max}}^p$ on smooth, compact manifolds with smooth boundary. This result is called Friedrichs’ Extension Lemma (Theorem 3.1). In this paper, we observe that the two notions of boundary values coincide, as well (Theorem 3.3). We call this Friedrichs’ Extension Lemma with boundary values. We believe that this is a quite useful result, because it allows to approximate weak boundary values (in the sense of distributions) by smooth sections. So, one can work in the $C^\infty$-category. We exploit this principle for the operator $Q = \overline{\partial}$ in order to derive the Bochner-Martinelli-Koppelman formula for $L^p$-forms with weak $\overline{\partial}$-boundary values (Theorem 5.6).

Weak $\overline{\partial}$-boundary values in the sense of distributions (Definition 4.1) are a classical subject of complex analysis (see Theorem 4.2, for example) and closely related to the investigation of the so-called Hardy spaces (cf. [Sk]). Starting from results of H. Skoda [Sk], F. R. Harvey, J. C. Polking [HaPo], U. Schuldenzucker [Sch] and T. Hefer [He2], there has been a major progress in the understanding of weak $\overline{\partial}$-boundary values made by T. Hefer in [He3], where boundary values in the sense of distributions are compared to boundary values which arise naturally in the application of integral operators. There, it is said that it is interesting to know under which hypothesis those types of boundary values agree, because the boundary values defined by restricting the kernel of an integral operator can often be estimated by direct methods, whereas the abstractly given distributional boundary values are less tractable but analytically interesting objects linked to the form on the interior of a domain. This is certainly true, for boundary values in the sense of distributions allow the application of a generalized Stokes’ formula, for example. Anyhow, one could think that boundary values in the sense of approximation (by $C^\infty$-forms) would be even more useful in many situations. That was the starting point and motivation for this article, which hopefully will contribute to the understanding of weak boundary values.
2. Weak Boundary Values

Let $M$ be a smooth, compact Riemannian manifold with smooth boundary, $E$ and $F$ Hermitian vector bundles over $M$, and

$$Q : C^\infty(M, E) \to C^\infty(M, F)$$

a differential operator of first order. Let $\sigma_Q$ be the principal symbol of $Q$, and

$$Q^* : C^\infty(M, F) \to C^\infty(M, E)$$

its formal adjoint operator given by

$$(Qu, v)_M = \int_M \langle Qu, v \rangle_F \, dV_M = \int_M \langle u, Q^* v \rangle_E \, dV_M = (u, Q^* v)_M,$$

where one of the two sections $u \in C^\infty(M, E)$, $v \in C^\infty(M, F)$ has compact support in the interior of $M$. Let $dS$ be the induced volume element on the boundary $bM$, $\nu$ the outward pointing unit normal to $bM$, and $\nu^\flat$ the dual cotangent vector. Then, the generalized Green-Stokes formula reads as (see [Ta], Prop. 9.1):

**Theorem 2.1.** Let $M$ be a smooth, compact Riemannian manifold with smooth boundary, and $Q$ a first-order differential operator (acting on sections of Hermitian vector bundles). Then

$$(Qu, v)_M - (u, Q^* v)_M = \frac{1}{i} \int_{bM} \langle \sigma_Q(x, \nu^\flat) u, v \rangle_F \nu_x \, dS(x)$$

for all sections $u \in C^\infty(M, E)$, $v \in C^\infty(M, F)$.

Now, let $u \in L^1(M, E)$ and $f \in L^1(M, F)$. Then we say that $Qu = f$ in the sense of distributions if

$$(u, Q^* \phi)_M = (f, \phi)_M$$

for all $\phi \in C^\infty(M, F)$ with compact support in the interior of $M$. We can now give the definition of weak boundary values with respect to the first-order differential operator $Q$:

**Definition 2.2.** In the situation of Theorem 2.1, let $u \in L^1(M, E)$ with $Qu \in L^1(M, F)$. Then $u$ has weak $Q$-boundary values $u_b \in L^p(bM, E|_{bM})$ if

$$(Qu, \phi)_M - (u, Q^* \phi)_M = \frac{1}{i} \int_{bM} \langle \sigma_Q(x, \nu^\flat) u_b, \phi \rangle_F \nu_x \, dS(x)$$ \hspace{1cm} (1)

for all $\phi \in C^\infty(M, F)$.

This generalizes the notion of weak boundary values of functions in the Sobolev space $H^{1,p}(M)$: Let $Q = d : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C} \otimes T^* M)$ be the exterior derivative. Then, for all $1 \leq p \leq \infty$, there is a unique continuous trace operator

$$T : H^{1,p}(M) = \{ u \in L^p(M, \mathbb{C}) : du \in L^p(M, \mathbb{C} \otimes T^* M) \} \to L^p(bM, \mathbb{C})$$

such that $Tu$ satisfies (1) (cf. [Alt], A 6.6). In general, weak $Q$-boundary values do not necessarily exist.
3. Friedrichs’ Extension Lemma

Again, let \( M \) be a smooth, compact Riemannian manifold with smooth boundary, \( E \) and \( F \) Hermitian vector bundles over \( M \), and \( Q : C^\infty(M, E) \to C^\infty(M, F) \) a first-order differential operator of first order. Let \( 1 \leq p < \infty \). Then, for \( f \in L^p(M, E) \), we say that \( f \in \text{dom}(Q^p_{\text{min}}) \) if there exists a sequence \( \{f_j\} \subset C^\infty(M, E) \) and \( g \in L^p(M, F) \) such that

\[
f_j \to f \text{ in } L^p(M, E), \quad Qf_j \to g \text{ in } L^p(M, F),
\]

and define \( Q^p_{\text{min}} f := g \). The operator \( Q^p_{\text{min}} \) is uniquely defined, because

\[
\langle g, h \rangle_M = \lim_{j \to \infty} \langle Qf_j, h \rangle_M = \lim_{j \to \infty} \langle f_j, Q^* h \rangle_M = \langle f, Q^* h \rangle_M
\]

for all \( h \in C^\infty(M, F) \) with compact support in the interior of \( M \). Moreover, we say that \( f \in \text{dom}(Q^p_{\text{max}}) \), if \( Qf = u \in L^p(M, F) \) in the sense of distributions, and set \( Q^p_{\text{max}} f := u \) in that case. It is easy to see that

\[
\text{dom}(Q^p_{\text{min}}) \subset \text{dom}(Q^p_{\text{max}}) \subset L^p(M, E),
\]

and \( Q^p_{\text{min}} \) is the restriction of \( Q^p_{\text{max}} \) to \( \text{dom}(Q^p_{\text{min}}) \). But, in our situation, also the converse is true (cf. [LiMi], Theorem V.2.6):

**Theorem 3.1. (Friedrichs’ Extension Lemma)** Let \( M \) be a smooth, compact Riemannian manifold with smooth boundary, and \( Q : C^\infty(M, E) \to C^\infty(M, F) \) a first-order differential operator (acting on sections of Hermitian vector bundles), and \( 1 \leq p < \infty \). Then for any \( f \in \text{dom}(Q^p_{\text{max}}) \) there exists a sequence \( \{f_\epsilon\} \) in \( C^\infty(M, E) \) such that \( \lim_{\epsilon \to 0} f_\epsilon = f \) and \( \lim_{\epsilon \to 0} Qf_\epsilon = Q^p_{\text{max}} f \) with respect to \( L^p \)-norms. Shortly this means that

\[
Q^p_{\text{min}} = Q^p_{\text{max}}.
\]

Let us recall the principles of the proof. Using a partition of unity, it is enough to consider \( U \subset \subset \mathbb{R}^n \) open with smooth boundary and \( Q : C^\infty(U) \to C^\infty(U) \). So, let \( f \in L^p(U) \) and \( Qf = Q^p_{\text{max}} f \in L^p(U) \). Again, by the partition of unity argument, one has to consider the following two cases:

1. \( \text{supp}(f) \subset \subset U \), or
2. \( U = \{x \in \mathbb{R}^n : x_1 < 0\} \) and \( \text{supp}(f) \subset \subset \overline{U} \).

For the first case, let \( \phi \in C^\infty_{\text{cpt}}(B_1(0)) \) with \( \phi \geq 0 \) and \( \int \phi dx = 1 \), where \( dx \) is the Euclidean volume element. We call \( \phi_\epsilon(x) := \epsilon^{-n} \phi(x/\epsilon) \) a Dirac sequence, and

\[
f_\epsilon := f * \phi_\epsilon
\]

the convolution of \( f \) with a Dirac sequence. It is well known that \( f_\epsilon \to f \) in \( L^p(U) \) for \( \epsilon \to 0^+ \). But the crucial observation is

**Lemma 3.2.**

\[
\|Qf_\epsilon - (Qf) * \phi_\epsilon\|_{L^p(U)} \lesssim \|f\|_{L^p(U)}.
\]
It is now easy to complete the first case: Let \( \delta > 0 \) and \( \psi \in C_\text{cpt}^\infty(U) \) such that
\[
\|f - \psi\|_{L^p(U)} < \delta.
\]
Applying Lemma 3.2 to \( f - \psi \) yields:
\[
\|Qf - (Qf)^*\|_{L^p(U)} \lesssim \delta + \|Q\psi - (Q\psi)^*\|_{L^p(U)}.
\]
Choosing \( \delta \) and \( \epsilon \) arbitrarily small finishes this part of the proof. The second case is treated by exactly the same procedure. One has only to be a little careful when choosing the Dirac sequence \( \phi_\epsilon \). Here, let \( \phi \in C_\text{cpt}^\infty(B_1(0)) \) such that
\[
\text{supp}(\phi) \subset \subset \{x \in B_1(0) : x_1 > 0\}.
\]
Then \( f_\epsilon \) is well defined on \( U \), Lemma 3.2 is still true and everything goes through as before. That completes the proof of Theorem 3.1 as it is given in [LiMi].

We are now interested in the behavior of the sequence \( \{f_\epsilon\} \) on the boundary \( bM \). It is possible to extend Theorem 3.1 to Friedrichs’ Extension Lemma with boundary values:

**Theorem 3.3.** In the situation of Theorem 3.1, assume that \( f \in \text{dom}(Q_{\text{max}}^p) \) has weak \( Q \)-boundary values \( f_b \in L^p(bM,E|_{bM}) \) in the sense of Definition 2.2. Then there exists a sequence \( \{f_\epsilon\} \) in \( C^\infty(M,E) \) such that \( \lim_{\epsilon \to 0} f_\epsilon = f \) in \( L^p(M,E) \), \( \lim_{\epsilon \to 0} Qf_\epsilon = Q_{\text{max}}^p f \) in \( L^p(M,F) \) and
\[
\lim_{\epsilon \to 0} \sigma_Q(\cdot,\nu^\dagger(\cdot))f_\epsilon|_{bM} = \sigma_Q(\cdot,\nu^\dagger(\cdot))f_b \quad \text{in} \quad L^p(bM,F|_{bM}).
\]

**Proof.** We copy the proof of Theorem 3.1. One has to be even more careful when choosing the Dirac sequence. We only have to take a closer look at the second case. So, let \( U = \{x \in \mathbb{R}^n : x_1 < 0\} \), \( \text{supp}(f) \subset U \) and
\[
\text{supp}(f_b) \subset \subset bU = \{x \in \mathbb{R}^n : x_1 = 0\}.
\]
Then
\[
Q = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j} + b(x), \quad Q^* = -\sum_{j=1}^n \left( a_j(x) \frac{\partial}{\partial x_j} + \frac{\partial a_j}{\partial x_j}(x) \right) + b(x),
\]
and
\[
\int_U (Qf)\Phi dx - \int_U f(Q^*\Phi) dx = \int_{bU} a_1(0,x') f_b(x')\Phi(0,x') dx'
\]
for all \( \Phi \in C_\text{cpt}^\infty(U) \) according to Definition 2.2 with \( x' = (x_2,\ldots,x_n) \). We will use the decomposition
\[
Q^* = -a_1(x) \frac{\partial}{\partial x_1} + Q'.
\]
Now, let us choose the right Dirac sequence for our purposes. Let $B_1'(0)$ be the unit ball in $\mathbb{R}^{n-1}$ and $\psi \in C^\infty_c(B_1'(0))$ with $\psi \geq 0$ and $\int \psi dx'$, where $dx'$ is the Euclidean volume element in $\mathbb{R}^{n-1}$, $x' = (x_2, ..., x_n)$. For $\epsilon > 0$ set
\[
\psi_\epsilon := \epsilon^{-(n-1)} \psi(x'/\epsilon).
\]
Therefore, it follows that
\[
\lim_{\epsilon \to 0} (a_1(0, \cdot) f_b) * \psi_\epsilon = a_1(0, \cdot) f_b \quad \text{in} \quad L^p(bU). \tag{4}
\]
Moreover, let $h : \mathbb{R} \to [0, 1]$ be a smooth function such that
\[
h(x_1) = \begin{cases} 
0, & \text{for } x_1 \leq 1, \\
1, & \text{for } x_1 \geq 2.
\end{cases}
\]
For $\tau > 0$, set $h_\tau(x_1) = h(x_1/\tau)$. Now, we define a Dirac sequence in $\mathbb{R}^n$:
\[
\phi_\epsilon(x) = \psi_\epsilon(x') h_\tau(\epsilon)(x_1),
\]
where $\tau(\epsilon)$ will be chosen later. At this point, we only require that $\tau(\epsilon) \leq \epsilon$. Let
\[
f_\epsilon := f * \phi_\epsilon.
\]
Then $f_\epsilon \to f$ and $Qf_\epsilon \to Qf$ as in the proof of Theorem 3.1. Because of (4), we only have to prove that
\[
\lim_{\epsilon \to 0} a_1(0, \cdot) f_\epsilon|_{bU} = \lim_{\epsilon \to 0} (a_1(0, \cdot) f_b) * \psi_\epsilon \quad \text{in} \quad L^p(bU). \tag{5}
\]
For $(0, x') \in bU$, we calculate:
\[
(a_1(0, \cdot) f_b) * \psi_\epsilon(x') = \int_{bU} a_1(0, t') f_b(t') \psi_\epsilon(x' - t') dt'
\]
\[
= \int_{bU} a_1(0, t') f_b(t') \psi_\epsilon(x' - t')(1 - h_\tau(\epsilon)(0)) dt'
\]
\[
= \int_U (Qf) \Phi_\epsilon x' dt - \int_U f(Q^* \Phi_\epsilon x') dt,
\]
if we let
\[
\Phi_\epsilon x'(t) := \psi_\epsilon(x' - t')(1 - h_\tau(\epsilon)(-t_1))
\]
and apply the Green-Stokes formula (2). By the use of the decomposition (3), it follows that
\[
(a_1(0, \cdot) f_b) * \psi_\epsilon(x') = \int_U (Qf) \Phi_\epsilon x' dt - \int_U f(Q^* \Phi_\epsilon x') dt
\]
\[
+ \int_U f(t) a_1(t) \phi_\epsilon(x - t) dt
\]
with $x = (0, x')$. We will now show that the first two terms turn to 0 in $L^p(bU)$ if we let $\epsilon \to 0$. 

So, consider:

\[
\int_{bU} \left| \int_U (Qf) \Phi_{x'}^\epsilon dt \right|^p dx' \leq \int_{bU} \int_U |Qf|^p (\Phi_{x'}^\epsilon)^p dt dx'
\]

\[
\leq \int_U |Qf|^p (1 - h_{\tau(\epsilon)}(-t_1)) \int_{bU} \psi(x' - t') dx' dt
\]

\[
= \int_U |Qf|^p (1 - h_{\tau(\epsilon)}(-t_1)) dt
\]

Here, \( |Qf|^p (1 - h_{\tau(\epsilon)}(-t_1)) \leq |Qf|^p \) (which is in \( L^1(U) \)), and converges to 0 point-wise. Hence, the whole expression turns to 0 by Lebesgue’s Theorem. For the second term, note that

\[
|Q_{x'}^\epsilon \Phi_{x'}^\epsilon| = \left| (1 - h_{\tau(\epsilon)}(-t_1)) Q_{x'}^\epsilon \psi_t(x' - \cdot) \right| \lesssim \epsilon^n (1 - h_{\tau(\epsilon)}(-t_1)).
\]

Hence, we conclude:

\[
\int_{bU} \left| \int_U f(Q_{x'}^\epsilon \Phi_{x'}^\epsilon) dt \right|^p dx' \leq \int_U |f|^p \left( \int_{bU} |Q_{x'}^\epsilon \Phi_{x'}^\epsilon|^p dx' \right) dt
\]

\[
= \int_U |f|^p \left( \int_{B^\epsilon(U')} |Q_{x'}^\epsilon \Phi_{x'}^\epsilon|^p dx' \right) dt
\]

\[
\lesssim \frac{1}{\epsilon} \int_U |f|^p (1 - h_{\tau(\epsilon)}(-t_1)) dt.
\]

Here now, for fixed \( \epsilon > 0 \), \( \epsilon^{-1} |f|^p \in L^1(U) \),

\[
\frac{1}{\epsilon} |f|^p (1 - h_{\tau(\epsilon)}(-t_1)) \leq \frac{1}{\epsilon} |f|^p,
\]

and the left-hand side converges to 0 point-wise for \( \tau \to 0 \). So, by the Theorem of Lebesgue, there exists \( \tau(\epsilon) \) such that

\[
\frac{1}{\epsilon} \int_U |f|^p (1 - h_{\tau(\epsilon)}(-t_1)) dt \leq \epsilon.
\]

This is our choice of \( \tau(\epsilon) \) which has been left open before. So, we have just seen that

\[
\lim_{\epsilon \to 0} (a_1(0, \cdot) f_b) * \psi_t(x') = \lim_{\epsilon \to 0} \int_U f(t) a_1(t) \phi_t(x - t) dt
\]

in \( L^p(bU) \). Recall that we had reduced the problem to showing (5). So, only

\[
\lim_{\epsilon \to 0} \int_U f(t) (a_1(t) - a_1(x)) \phi_t(x - t) dt = 0
\]

in \( L^p(bU) \) remains to show. But, due to compactness, there exists a constant \( C > 0 \) such that \( |a_1(t) - a_1(x)| \leq C \epsilon \) if \( |t - x| \leq \epsilon \). Since \( f \in L^p(U) \) and \( |\phi_t| \leq 1 \), the proof is finished easily. □

We remark that the assumptions on the regularity of the boundary \( bM \) could be relaxed considerably.
4. Boundary Values for the $\partial$-Operator

In this section, we will apply Friedrichs’ Extension Lemma with boundary values, Theorem 3.3, to the $\partial$-operator. Recall the following Definition of $\partial$-boundary values that is common in complex analysis:

**Definition 4.1.** Let $D \subset \subset \mathbb{C}^n$ be a bounded domain with smooth boundary $bD$, and $f \in L^p_{0,q}(D)$ with $\partial f \in L^p_{0,q+1}(D)$ in the sense of distributions for $1 \leq p < \infty$. Then, we say that $f$ has weak $\partial$-boundary values $f_b \in L^p(bD)$ if

$$\int_D \partial f \wedge \phi + (-1)^q \int f \wedge \partial \phi = \int_{bD} f_b \wedge \iota^*(\phi)$$

(6)

for all $\phi \in C^\infty_{n,q-1}(\overline{D})$, where $\iota : bD \hookrightarrow \mathbb{C}^n$ is the embedding of the boundary.

In fact, the left hand side of (6) depends only on the pull-back $\iota^*(\phi)$ of $\phi$ to $bD$, and so it defines a current on $bD$. Generally, this current is called the weak $\partial$-boundary value of $f$, and we say that $f$ has got boundary values in $L^p$, if this current can be represented by an $L^p$-form as in Definition 4.1. See [He2] for a more detailed treatment of that topic. Boundary values as in Definition 4.1 are not uniquely defined. The reason is as follows: Let $r \in C^\infty(\mathbb{C}^n)$ be a defining function for $D$. So, $D = \{ z \in \mathbb{C}^n : r(z) < 0 \}$ and we may assume that $\|dr\| \equiv 1$ on $bD$. Then $\iota^*(dr) = 0$ implies $\iota^*(\partial r) = -\iota^*(\partial r)$. Note that $\phi \in C^\infty_{n,n-q-1}(\overline{D})$ contains $\partial r$ necessarily. Hence, $\iota^*(\partial r) \wedge \iota^*(\phi) = 0$ on $bD$ for all $\phi \in C^\infty_{n,n-q-1}(\overline{D})$. One should mention an example where weak $\partial$-boundary values occur:

**Theorem 4.2. (Harvey-Polking [HaPo])** Let $r$ be the strictly plurisubharmonic defining function of a strictly pseudoconvex domain $D \subset \subset \mathbb{C}^n$, and $\omega \in L^1_{0,1}(D)$ with $\partial \omega = 0$ and $|r|^{-1/2} \partial r \wedge \omega \in L^1_{0,2}(D)$. Then there exists $f \in L^1(D)$ with weak $\partial$-boundary values $f_b \in L^1(bD)$ such that $\partial f = \omega$.

We will now show that Definition 4.1 is actually equivalent to Definition 2.2, if we make the right choices. So, let $M = \overline{D}$ with the underlying Riemannian structure on $\mathbb{C}^n$, $E = \Lambda^0 q T^* M$, $F = \Lambda^{0,q+1} T^* M$, and

$$Q := \partial : C^\infty_{0,q}(M) = C^\infty(M,E) \longrightarrow C^\infty_{0,q+1}(M) = C^\infty(M,F).$$

Note that $Q^* = - \ast \partial \ast$. For $u, v \in C^\infty_{0,q}(\overline{D})$:

$$(u, v)_M = \int_M \langle u, v \rangle_{E} dV_{\mathbb{C}^n} = \int_M u \wedge \ast \overline{v}.$$ 

In order to reformulate (6), let $g := (-1)^{q+1} \ast \phi \in C^\infty_{0,q+1}(D)$. Then $\phi = \ast \overline{g}$. So,

$$\int_D \partial f \wedge \phi = \int_D \partial f \wedge \ast \overline{g} = \int_M \langle \partial f, g \rangle_F dV_M = (Qf, g)_M,$$
and
\[
\int_D f \wedge \overline{\partial} \phi = - \int_D f \wedge * * \overline{\partial} * * \phi = \int_D f \wedge * Q^* \overline{\partial} \\
= (-1)^{q+1} \int_M (f, Q^* g)_E \, dV_{\mathbb{C}^n} = (-1)^{q+1} (f, Q^* g)_M.
\]

Hence, in the notation of Definition 2.2, the left hand side of (6) reads exactly as \((Qf, g)_M - (f, Q^* g)_M\). For the right hand side, recall that we have chosen the defining function \(r\) such that \(\|dr\| \equiv 1\) on \(bD\). That implies \(dS_{bD} = \iota^*(dr)\).

Note that there is a \((0, q)\)-form \(f'_b \in C^\infty(bD, \Lambda^0T^*\mathbb{C}^n|_{bD})\) such that \(\iota^*(f'_b) = f_b\). Since \(\iota^* dr = 0\) and \(dr \wedge \phi = \overline{\partial} r \wedge \phi\), we compute
\[
f_b \wedge \iota^*(\phi) = \iota^*(f'_b \wedge \phi) = \iota^*((dr \wedge f'_b \wedge \phi) \wedge \phi) = (\overline{\partial} r \wedge f'_b \wedge \phi) \, dS_{bD}
\]
\[
= (\overline{\partial} r \wedge f'_b \wedge \overline{\partial} f'_b) \, dS_{bD} = (\overline{\partial} r \wedge f'_b, g)_F \, dS_{bD}
\]
\[
= (Q(r f'_b), g)_F \, dS_{bD} = \frac{1}{i} (\sigma_Q(\cdot, \nu^b) f'_b, g)_F \, dS_{bD}.
\]

So, we have
\[
\int_{bD} f_b \wedge \iota^*(\phi) = \frac{1}{i} \int_{bM} \langle \sigma_Q(x, \nu^b) f'_b, g \rangle_F \, dS_{bM}(x),
\]
and recognize therefore:

**Lemma 4.3.** If \(f \in L^p_{0,q}(D)\) with \(\overline{\partial} f \in L^p_{0,q+1}(D)\) has weak \(\overline{\partial}\)-boundary values \(f_b \in L^p_b(bD)\) according to Definition 4.1 exactly if it has \(\overline{\partial}\)-boundary values \(f'_b \in L^p(bD, \Lambda^0T^*\mathbb{C}^n|_{bD})\) according to Definition 2.2.

So, we are now in the position to translate Theorem 3.3 into the Friedrichs’ Extension Lemma with boundary values for the \(\overline{\partial}\)-operator:

**Theorem 4.4.** Let \(D \subset \subset \mathbb{C}^n\) be a bounded domain with smooth boundary \(bD\), and \(f \in L^p_{0,q}(D)\) with \(\overline{\partial} f \in L^p_{0,q+1}(D)\) in the sense of distributions for \(1 \leq p < \infty\). Then \(f\) has weak \(\overline{\partial}\)-boundary values \(f_b \in L^p_b(bD)\) according to Definition 4.1 exactly if there is a sequence \(\{f_\epsilon\}\) in \(C_0^\infty(D)\) such that \(\lim_{\epsilon \to 0} f_\epsilon = f\) in \(L^p_{0,q}(D)\), \(\lim_{\epsilon \to 0} \overline{\partial} f_\epsilon = \overline{\partial} f\) in \(L^p_{0,q+1}(D)\), and

\[
\lim_{\epsilon \to 0} \iota^*(f_\epsilon \wedge \phi) = f_b \wedge \iota^*(\phi) \quad \text{in} \; L^p_{2n-1}(bD)
\]

for all \(\phi \in C_0^\infty(\mathbb{C}^n)\). If \(r \in C^\infty(\mathbb{C}^n)\) is a defining function for \(D\), i.e. \(D = \{z \in \mathbb{C}^n : r(z) < 0\}\) and \(dr \neq 0\) on \(bD\), then the last condition is equivalent to

\[
\lim_{\epsilon \to 0} \iota^*(f_\epsilon \wedge \partial r) = f_b \wedge \iota^*(\partial r) \quad \text{in} \; L^p_{q+1}(bD).
\]

If \(q = 0\), then this in turn is equivalent to

\[
\lim_{\epsilon \to 0} \iota^*(f_\epsilon) = f_b \quad \text{in} \; L^p(bD).
\]
5. Regularity of the BMK Formula

The characterization of weak $\partial$-boundary values by approximation is a quite useful tool because it allows us to simply work in the $C^\infty$-category in many situations. As an application, we will derive the Bochner-Martinelli-Koppelman formula for $L^p$-forms with weak $\partial$-boundary values. Before doing that, we present another technical but useful result. For convenience, let us recall shortly the Bochner-Martinelli-Koppelman formula.

Definition 5.1. Let $0 \leq q \leq n$. The Bochner-Martinelli-Koppelman kernel $B_{nq}$ in $\mathbb{C}^n$ is then given as

$$B_{nq}(\zeta, z) = \frac{(n-1)!}{2^{q+1} \pi^n} \frac{1}{\|\zeta - z\|^2n} \sum_{j,J, |L| = q+1} e^A_j \epsilon_A B_{nq}(\zeta_j - z_j)(\ast \text{d}\zeta^L) \wedge \text{d}z^J,$$

where

$$\epsilon_A^B := \begin{cases} \text{sign } \pi, & \text{if } A = B \text{ as sets and } \pi \text{ is a permutation with } B = \pi A, \\ 0, & \text{if } A \neq B. \end{cases}$$

Moreover, let $B_{n,-1} \equiv 0$.

Now, let $D \subset \subset \mathbb{C}^n$ be a bounded domain with $C^1$-smooth boundary $bD$. If $g$ is a measurable $(0, q+1)$-form on $D$, let

$$B^D_q g(z) := \int_D g(\zeta) \wedge B_{nq}(\zeta, z),$$

and if $f$ is a measurable $q$-form on $bD$, let

$$B^{bD}_q f(z) := \int_{bD} f(\zeta) \wedge B_{nq}(\zeta, z),$$

provided, the integrals do exist. Then:

Theorem 5.2. (BMK formula [Ko]) Let $D \subset \subset \mathbb{C}^n$ be a bounded domain with $C^1$-smooth boundary $bD$, $1 \leq q \leq n$, and $f \in C^1_{0,q}(\overline{D})$. Then:

$$f(z) = B^{bD}_q f(z) - B^D_q (\overline{\partial} f)(z) - \overline{\partial}_z B^{bD}_{q-1} f(z), \quad (7)$$

where $B^{bD}_{q-1} f \in C^1_{0,q-1}(D)$.

In the following, we will show that (7) is still valid under the assumption that $f \in L^1_{0,q}(D)$ with $\overline{\partial} f \in L^1_{0,q+1}(D)$ has weak $\partial$-boundary values $f_b \in L^1_q(bD)$. It is well-known that

Lemma 5.3. Let $D \subset \subset \mathbb{C}^n$ be a bounded domain. Then, $B^D_q$ defines a bounded linear operator

$$L^p_{0,q+1}(D) \to L^r_{0,q}(D)$$

for all $1 \leq p, r \leq \infty$ such that $1/r > 1/p - 1/(2n)$. 
This is a direct consequence of \(|B_{nq}(\zeta,z)| \lesssim |\zeta - z|^{2n-1}\) and Young’s inequality, which is usually used for estimating integral operators (cf. for example [LiMi], Proposition III.5.35). In order to estimate the BMK boundary operator \(B_{bD}^q\), we need a more general version of such an inequality. So, we will make use of the following result. The proof can be found in [Ru2], Theorem 3.3.4:

**Theorem 5.4.** Let \(1 \leq t \leq s < \infty\) and \(1 \leq a,b \leq \infty\) be fixed, \((X,\mu)\) and \((Y,\nu)\) measure spaces with \(\mu(X) < \infty\) and \(\nu(Y) < \infty\), and \(K\) a \(\mu \times \nu\)-measurable function on \(X \times Y\) such that

\[
\int_X |K(x,y)|^t d\mu(x) \leq g(y) \quad \text{for almost all } y \in Y, \\
\int_Y |K(x,y)|^s d\nu(y) \leq h(x) \quad \text{for almost all } x \in X,
\]

where \(g \in L^a(Y)\) and \(h \in L^b(X)\). Then:

I. The linear operator \(f \mapsto Tf\) which is given by

\[
Tf(y) = \int_X K(x,y)f(x)d\mu(x)
\]

for almost all \(y \in Y\) defines a bounded operator \(T : L^p(X) \to L^r(Y)\) for all \(1 \leq p,r \leq \infty\) satisfying

\[
p \geq \left\{ \begin{array}{ll}
\frac{t}{t-1} & , \text{if } t > 1, \\
\infty & , \text{if } t = 1,
\end{array} \right.
\]

and

\[
r \leq at.
\]

II. The mapping \(f \mapsto Tf\) is bounded as an operator \(T : L^p(X) \to L^1(Y)\) for \(1 \leq p < \infty\) with

\[
p \geq \left\{ \begin{array}{ll}
\frac{sb}{sb-t} & , \text{if } 1 < sb < \infty, \\
1 & , \text{if } b = \infty.
\end{array} \right.
\]

III. If (11) is satisfied and \(sb \neq t\), then \(f \mapsto Tf\) defines a bounded operator \(T : L^p(X) \to L^r(Y)\) for all \(1 \leq r \leq \infty\) with

\[
\frac{1}{r} = \left(\frac{sb}{sb-t}\right)\left(\frac{1}{p} + \frac{1}{t} - 1\right)
\]

and

\[
r \leq t \left(\frac{a-s-t}{s} + 1\right).
\]

We have made the following conventions: In (12), let \(1/r = 0\) if \(r = \infty\). If \(b = \infty\), then (12) has to be interpreted as \(\frac{1}{r} = \frac{1}{p} + \frac{1}{t} - 1\). If \(a = \infty\), then (13) reads as \(r \leq \infty\).
Lemma 5.5. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain with $C^1$-smooth boundary $bD$. Then, $B_q^{bD}$ is bounded as an operator

$$L_q^p(bD) \to L_{0,q}^p(D)$$

for all $1 \leq p < \infty$.

Proof. We will apply Theorem 5.4 to the operator $B_q^{bD}$. So, let $X = bD$, $Y = D$ and $|K(x,y)| = |B_{nq}(x,y)| \leq \frac{A}{|x-y|^{2n-1}}$, where $A > 0$ is a constant that depends only on $D$, $q$ and $n$. We choose $t = 1$.

It is not hard to prove that there are constants $C_0(D) > 0$ and $C_1(D) > 0$ such that

$$\int_X |K(x,y)|^s d\mu(x) \leq C_0(D) + C_1(D) |\log \delta(y)| =: g(y),$$

where $\delta(y) := \text{dist}(y,bD)$. For a proof, we refer to [Ru2], Lemma 3.3.1. It is easy to see that $|g|^n$ is integrable over $Y = D$ for all powers $1 \leq a < \infty$. So, we remark that $g \in L^a(Y)$ for all $1 \leq a < \infty$ (cf. [Ru2], Lemma 3.3.3). Now, choose $s > 1$ such that

$$1 = t < s < \frac{2n}{2n-1}.$$

Then

$$h(x) := \int_Y |K(x,y)|^s d\nu(y)$$

is uniformly bounded (independent of $x \in X$). Hence $h \in L^\infty(X)$. So, the assumptions of Theorem 5.4 are fulfilled for $X = bD$, $Y = D$, $T = B_q^{bD}$, $1 = t < s$, $h \in L^\infty(X)$, i.e. $b = \infty$, and $g \in L^a(Y)$ for all $1 \leq a < \infty$. We conclude that $B_q^{bD}$ defines a bounded linear operator $B_q^{bD} : L_q^p(bD) \to L_{0,q}^r(D)$ for all $1 \leq p, r < \infty$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{t} - 1 = \frac{1}{p}.$$

We have now provided all the tools that are needed to derive the Bochner-Martinelli-Koppelman formula for $L^p$-forms with weak $\overline{\partial}$-boundary values as an application of Friedrichs’ Extension Theorem with boundary values. So, let $D \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary, $1 \leq r, p < \infty$, $f \in L_{0,q}^p(D)$ with $\overline{\partial} f \in L_{0,q+1}^r(D)$ in the sense of distributions and assume that $f$ has weak $\overline{\partial}$-boundary values $f_b \in L_q^p(bD)$ according to Definition 4.1.
Then, by Theorem 4.4, it follows that there exists a sequence \( \{f_\epsilon\} \) in \( C_0^\infty(\overline{D}) \) such that
\[
\lim_{\epsilon \to 0} f_\epsilon = f \quad \text{in} \quad L^1_{0,q}(D),
\]
\[
\lim_{\epsilon \to 0} \partial f_\epsilon = \partial f \quad \text{in} \quad L^1_{0,q+1}(D),
\]
and
\[
\lim_{\epsilon \to 0} f_\epsilon \big|_{bD} = \lim_{\epsilon \to 0} \partial f_\epsilon = \partial f \quad \text{in} \quad L^1_{0,q-1}(D).
\]

For all \( \phi \in C_\infty_{n,n-q-1}(\overline{D}) \), where \( \iota : bD \to \mathbb{C}^n \) denotes the embedding. In the following, we will simply write \( f_\epsilon \) instead of \( f_\epsilon \big|_{bD} \).

Now, the classical BMK formula, Theorem 5.2, implies:
\[
f_\epsilon(z) = B_{bD}^{q}(f_\epsilon(z)) - B_{D}^{q}(\partial f_\epsilon)(z) - \partial z B_{D}^{q-1} f_\epsilon(z)
\]
for all \( z \in D \), which we permute to:
\[
\partial z B_{D}^{q-1} f_\epsilon(z) = B_{bD}^{q} f_\epsilon(z) - B_{D}^{q}(\partial f_\epsilon)(z) - f_\epsilon(z).
\]

By Lemma 5.3 and Lemma 5.5, we know that the applications
\[
B_{bD}^{q} : L^1_q(bD) \to L^1_{0,q}(D),
\]
\[
B_{D}^{q} : L^1_{0,q+1}(D) \to L^1_{0,q}(D)
\]
are continuous. Hence, the right hand side of (15) converges in \( L^1_{0,q}(D) \) to a form
\[
G = B_{bD}^{q} f_\epsilon - B_{D}^{q}(\partial f_\epsilon) - f \in L^1_{0,q}(D).
\]

To see this, note that the Bochner-Martinelli-Koppelman kernel \( B_{nq}(\zeta, z) \) is a \((n, n-q-1)\)-form in \( \zeta \). So, (14) can be used.

Since
\[
\lim_{\epsilon \to 0} B_{D}^{q-1} f_\epsilon = B_{D}^{q-1} f \quad \text{in} \quad L^1_{0,q-1}(D),
\]
\[
\lim_{\epsilon \to 0} \partial z B_{D}^{q-1} f_\epsilon = G \quad \text{in} \quad L^1_{0,q}(D),
\]
it follows that \( G \) actually is the \( \overline{\partial} \)-derivate in the sense of distributions:
\[
G = \partial z B_{D}^{q-1} f.
\]

Applying Lemma 5.3 and Lemma 5.5 again, we observe that
\[
B_{bD}^{q} f_\epsilon \in L^p_{0,q}(D),
\]
\[
B_{D}^{q}(\partial f) \in L^r_{0,q}(D).
\]

So, the right hand side of (16), and therefore \( G \), is in \( L^r_{0,q}(D) \cap L^p_{0,q}(D) \).
We summarize:

**Theorem 5.6. (BMK formula for $L^p$-forms)** Let $D \subset \subset \mathbb{C}^n$ be a bounded domain with smooth boundary and $0 \leq q \leq n$. Moreover, let $1 \leq r, p < \infty$ and $f \in L^p_{0,q}(D)$ with $\bar{\partial}f \in L^r_{0,q+1}(D)$, such that $f$ has weak $\partial$-boundary values $f_b \in L^p_{q}(bD)$. Then

\[
\begin{align*}
B^b_{q} f_b &\in L^p_{0,q}(D), \\
B_{q}^D(\bar{\partial}f) &\in L^r_{0,q}(D), \\
B^{\bar{\partial}}_{q-1} f &\in L^p_{0,q-1}(D) \cap \text{Dom}(\bar{\partial}), \\
\bar{\partial}B^{\bar{\partial}}_{q-1} f &\in L^p_{0,q}(D) \cap L^p_{0,q}(D),
\end{align*}
\]

and

\[
f(z) = B^b_{q} f_b(z) - B_{q}^D(\bar{\partial}f)(z) - \bar{\partial}B^{\bar{\partial}}_{q-1} f(z)
\]

for almost all $z \in D$.

**References**


Mathematisches Institut, Universität Bonn, Beringstr. 1, D-53115 Bonn, Germany.

E-mail address: jean@math.uni-bonn.de